NOT-brainstorming session, Leuven, June 11th 2012

Duality and reversibility: squares versus crosses

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0.1 Aims and claims of the talk

=> integrate three lines of recent inquiry:

1. duality in Piaget & Gottschalk 'PG' (NOT workshop, Nice 2010)
2. algebra for Set Inclusion (Alessio, Nice 2011)
3 "hybrid' geometries (Lorenz, Nancy 2011/Bochum 2012)

=> Present the Set Inclusion relations/Algebra as a new decoration for β3, the tetrahexahedron (THH), or the rhombic dodecahedron (RDH)

1. two 'classical' decorations propositional connectives
   S5 modalities
2. two recent decorations
   Public Announcement Logic (Lorenz, Corte 2010)
   Sherwood-Czezowski singular propositions (LNAT2, Diagrams)

=> Use the Set Inclusion Algebra to argue that the PG Duality Geometry is (1) hybrid and (2) degenerate, and propose two 'solutions':

(1) decompose the notion of duality into different geometries
(2) distinguish duality squares from duality crosses

=> Decompose Duality geometry in two steps:

1. distinction between
   syntax: operations on formulae => **Reversibility Geometry**
   semantics: operations on bitstrings

2. within operations on bitstrings distinction between
   ‘horizontal’ mirroring operations => **Flip Geometry**
   ‘vertical’ polarity operations => **Switch Geometry**

=> Switch Geometry **<= Duality Geometry =>** Flip Geometry
4 squares 2 squares 1 square

Switch Geom **>? Reversibility Geom >?** Duality Geom > Flip Geom
4 squares 4/3/2 squares 2 squares 1 square

0.2 Overview of the talk

1 The Logical Geometry of Set Inclusion

1.1 The Set Inclusion tripartition
1.2 The eight standard Set Inclusion relations
1.3 The Boolean closure of Set Inclusion
1.4 Set Inclusion in RDH and beyond

2 Duality and reversibility in the Set Inclusion Algebra

2.1 Duality in Piaget and Gottschalk
2.2 Duality and symmetry
2.3 Duality geometry: squares and crosses
2.4 Aristotelian geometry: squares and crosses
2.5 Reversibility geometry: squares
2.6 Flip geometry: squares, bars and loops
2.7 Switch geometry: squares
2.8 The hybrid nature of the Duality Geometry
2.9 Duality and reversibility with the Propositional Connectives

3 Duality squares and crosses from 2D to 3D

3.1 Duality vs Aristotelian relations in square/hexagon
3.2 Aristotelian relations in RDH
3.3 Duality relations in 2D: Piaget
3.4 Duality relations in 3D: RDH

4 Conclusion and prospects

1 The Logical Geometry of Set Inclusion

1.1 The Set Inclusion tripartition

"the problem": cfr. Alessio:

strict order relations: $<$, $>$, $\leq$, $\geq$ ($=$, $\neq$)

yield an Aristotelian square and a Sesmat-Blanché hexagon

Set Inclusion relations $\subset$, $\supset$, $\subseteq$, $\supseteq$,($=$, $\neq$)

do NOT yield a similar Aristotelian square nor Sesmat-Blanché hexagon

"the cause": Set Inclusion works with 8 basic relations instead of 4/6:

$\subset$, $\supset$, $\subseteq$, $\supseteq$, $\not\subset$, $\not\supset$, $\not\subseteq$, $\not\supseteq$

"the aim": Establish the Logical Geometry of Set Inclusion

"the two steps":

=> define the semantics of the eight basic relations of Set Inclusion

=> define the Boolean closure of these eight relations

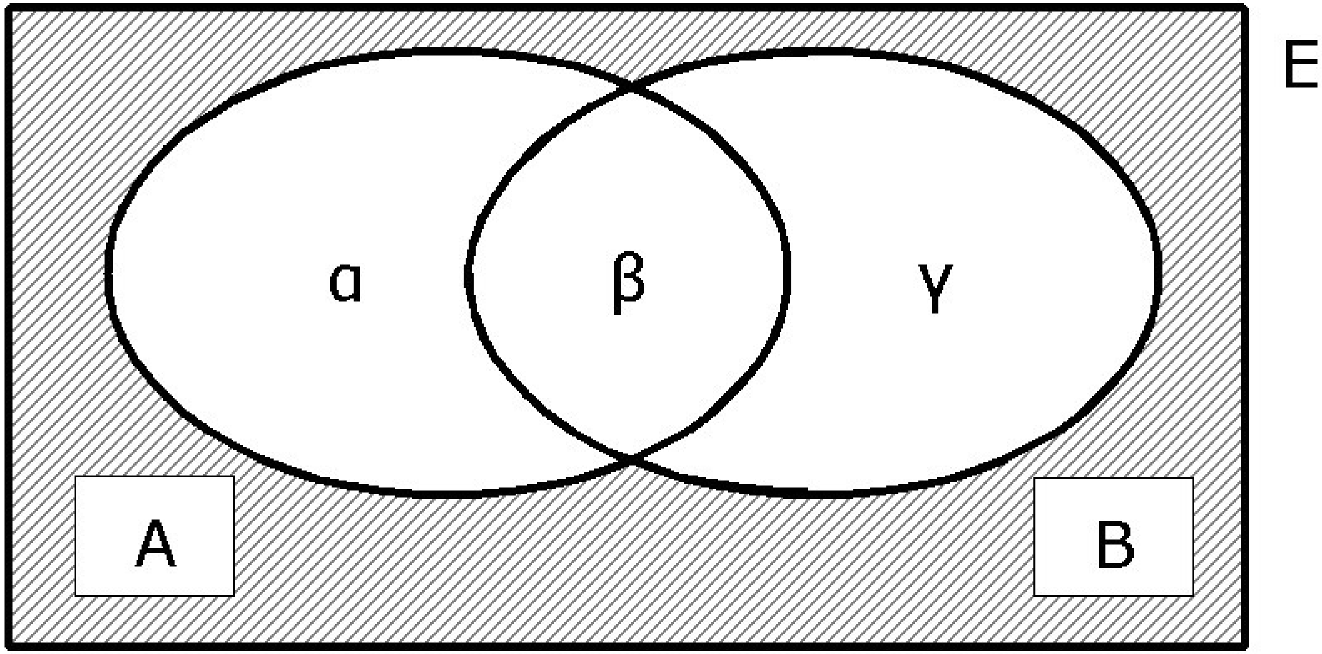


Figure 1 *Set Inclusion tripartition*

3 areas => 3 conditions on emptiness

=> area outside $A \cup B$ is irrelevant for Set Inclusion:

(1) $[\alpha = 0] \equiv A \setminus B = \varnothing$ $[\alpha = 1] \equiv A \setminus B \neq \varnothing$

$[\beta = 0] \equiv A \cap B = \varnothing$ $[\beta = 1] \equiv A \cap B \neq \varnothing$

$[\gamma = 0] \equiv B \setminus A = \varnothing$ $[\gamma = 1] \equiv B \setminus A \neq \varnothing$

p^q semantics: 3 questions, one for each area, with 2 answers, empty or not
$2^3 = 8$ constellations of combinations of 3 answers

	α	β	γ	
Σ1	0	0	0	[A = ∅] & [B = ∅]
Σ2	0	0	1	[A = ∅]
Σ3	0	1	0	‘mutual inclusion’
Σ4	0	1	1	‘proper inclusion left-to-right’
Σ5	1	0	0	[B = ∅]
Σ6	1	0	1	‘mutual exclusion’
Σ7	1	1	0	‘proper inclusion right-to-left’
Σ8	1	1	1	‘no inclusion, no mutual exclusion’

Table 1 *Eight State descriptions for the Set Inclusion tripartition*

grey rows: 3 out of the 8 states are ‘trivial’:at least one of the sets is empty => the 5 ***Gergonne*** relations (cfr. Ferdinando/Alessio)

1.2 The eight standard Set Inclusion relations

Interpret 8 states as combinations of truth-values = rows in truth table
Define the 8 relations of Set Inclusion in terms of columns in truth table: in which of the 8 states (including the trivial ones) does a given relation hold?

	αβγ	R1 $A \subset B$	R2 $A \not\subset B$	R3 $A \subseteq B$	R4 $A \not\subseteq B$	R5 $A \supset B$	R6 $A \not\supset B$	R7 $A \supseteq B$	R8 $A \not\supseteq B$
Σ1	000	0	1	1	0	0	1	1	0
Σ2	001	1	0	1	0	0	1	0	1
Σ3	010	0	1	1	0	0	1	1	0
Σ4	011	1	0	1	0	0	1	0	1
Σ5	100	0	1	0	1	1	0	1	0
Σ6	101	0	1	0	1	0	1	0	1
Σ7	110	0	1	0	1	1	0	1	0
Σ8	111	0	1	0	1	0	1	0	1

Table 2 *Truth Table for the 8 basic relations of Set Inclusion*

grey rows: reduction from eight to four bits ~ redundancy
whether or not the intersection area is empty is irrelevant for the semantics of Set Inclusion:

(2) $[v(\Sigma 1) = v(\Sigma 3)]$ & $[v(\Sigma 2) = v(\Sigma 4)]$ & $[v(\Sigma 5) = v(\Sigma 7)]$ & $[v(\Sigma 6) = v(\Sigma 8)]$

(3)	R1	$A \subset B$	01**01**00**00**	=>	**0100**
	R2	$A \not\subset B$	10**10**11**11**	=>	**1011**
	R3	$A \subseteq B$	11**11**00**00**	=>	**1100**
	R4	$A \not\subseteq B$	00**00**11**11**	=>	**0011**
	R5	$A \supset B$	00**00**10**10**	=>	**0010**
	R6	$A \not\supset B$	11**11**01**01**	=>	**1101**
	R7	$A \supseteq B$	10**10**10**10**	=>	**1010**
	R8	$A \not\supseteq B$	01**01**01**01**	=>	**0101**

negation relations = reversal of value in all positions

entailment relations: double lattice: 'gamma'-structure in NOT (Moretti)
central symmetry around black dot = negation

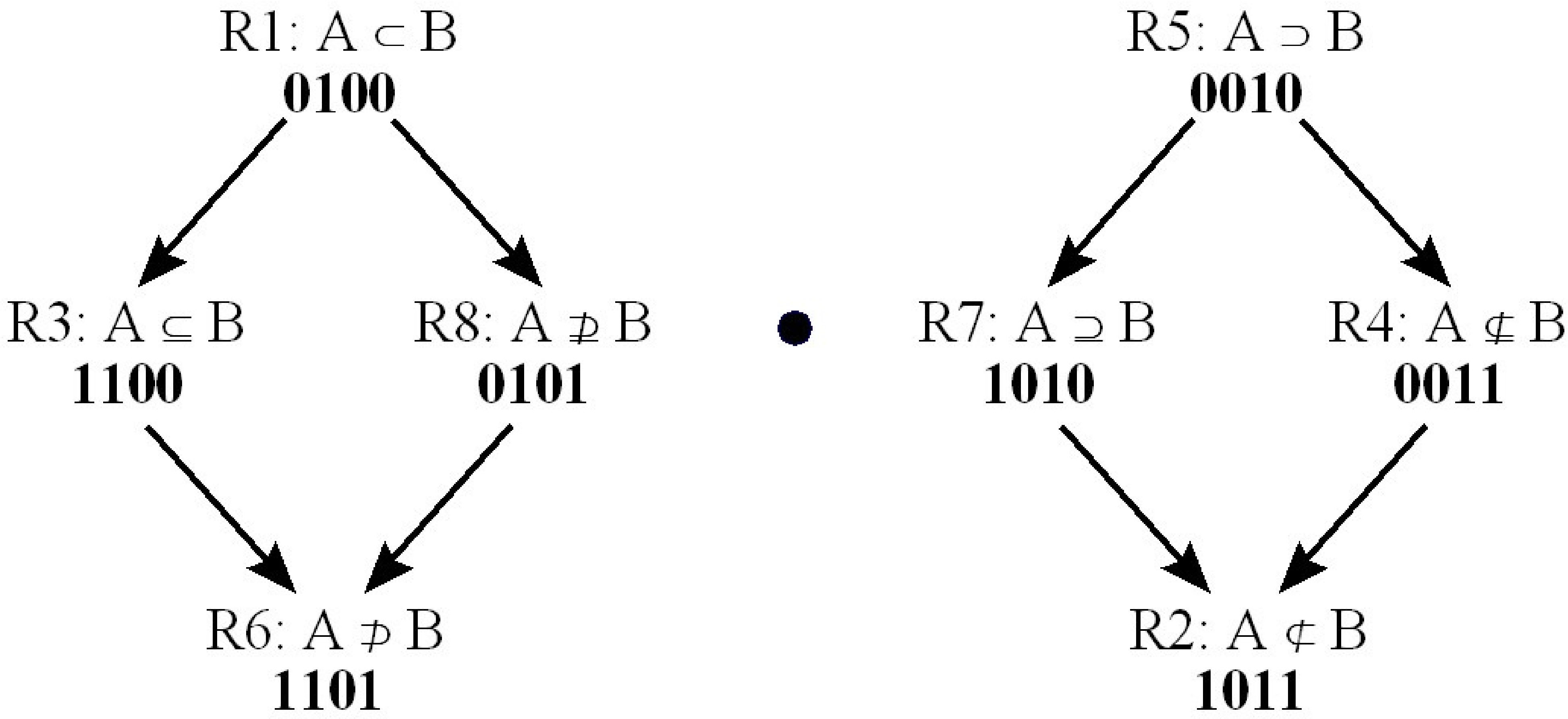


Figure 2 *Entailment relations between the 8 basic relations of Set Inclusion*

1.3 The Boolean closure of Set Inclusion

Boolean combinations among 4 members of 1 lattice = trivial entailment: join is smallest, meet is biggest

Non-trivial = check meet and join between each of 4 relations in left lattice and each of 4 relations in right lattice => 4 x 4 x 2 = 32

left lattice in Figure 2 = 4 rows of Tables 3/4, right lattice in Figure 2 = 4 colums of Tables 3/4

$\wedge$	$A \supset B$ 0010	$A \supseteq B$ 1010	$A \nsubseteq B$ 0011	$A \not\subset B$ 1011
$A \subset B$ 0100	0000	0000	0000	0000
$A \subseteq B$ 1100	0000	1000	0000	1000
$A \nsupseteq B$ 0101	0000	0000	0001	0001
$A \not\supset B$ 1101	0000	1000	0001	1001

$\vee$	$A \supset B$ 0010	$A \supseteq B$ 1010	$A \nsubseteq B$ 0011	$A \not\subset B$ 1011
$A \subset B$ 0100	0110	1110	0111	1111
$A \subseteq B$ 1100	1110	1110	1111	1111
$A \nsupseteq B$ 0101	0111	1111	0111	1111
$A \not\supset B$ 1101	1111	1111	1111	1111

Table 3 *Non-trivial meet/join operations on the 8 basic relations of Set Inclusion*

6 non-trivial cases:	2 equiv. classes of 1 green formula (=2)	(4a-b)
	2 equiv. classes of 3 yellow formulae (= 6)	(5a-b)
	2 equiv. classes of 3 orange formulae (= 6)	(5c-d)
2 trivial cases:	2 equiv. classes of 9 red formulae (= 18)	(6a-b)

NOTE: introduce conventions:

1. Boolean combinations have left-to-right Inclusion relation as their left conjunct and the right-to-left Inclusion relation as their right conjunct ('mirror image').
2. If both relations have same 'direction' then the relation without equality is the left conjunct.

(4)	a. R9	**($A \subset B$)** $\vee$ **($A \supset B$)**	$0100 \vee 0010 =$ **0110**	
	b. R10	**($A \not\subset B$)** $\wedge$ **($A \not\supset B$)**	$1011 \wedge 1101 =$ **1001**	

(5)	a. R11a	**($A \subseteq B$)** $\wedge$ **($A \supseteq B$)**	$1100 \wedge 1010 =$ **1000**
	R11b	($A \not\subset B$) $\wedge$ ($A \subseteq B$)	$1011 \wedge 1100 = 1000$
	R11c	($A \not\supset B$) $\wedge$ ($A \supseteq B$)	$1101 \wedge 1010 = 1000$
	=>	three equivalent ways of defining Mutual Inclusion	
	b. R12a	**($A \not\subseteq B$)** $\vee$ **($A \not\supseteq B$)**	$0011 \vee 0101 =$ **0111**
	R12b	($A \subset B$) $\vee$ ($A \not\subseteq B$)	$0100 \vee 0011 = 0111$
	R12c	($A \supset B$) $\vee$ ($A \not\supseteq B$)	$0010 \vee 0101 = 0111$
	=>	three equivalent ways of defining non-Mutual-Inclusion	

c. R13a **(A $\nsubseteq$ B)** $\wedge$ **(A $\nsupseteq$ B)** 0011 $\wedge$ 0101 = **0001**
R13b (A $\not\subset$ B) $\wedge$ (A $\nsupseteq$ B) 1011 $\wedge$ 0101 = 0001
R13c (A $\nsubseteq$ B) $\wedge$ (A $\not\supset$ B) 0011 $\wedge$ 1101 = 0001
=> three equivalent ways of defining Non-Inclusion

d. R14a **(A $\subseteq$ B)** $\vee$ **(A $\supseteq$ B)** 1100 $\vee$ 1010 = **1110**
R14b (A $\subset$ B) $\vee$ (A $\supseteq$ B) 0100 $\vee$ 1010 = 1110
R14c (A $\subseteq$ B) $\vee$ (A $\supset$ B) 1100 $\vee$ 0010 = 1110
=> three equivalent ways of defining 'non-Non-Inclusion'

(6) a. R15 **(A $\subset$ B)** $\wedge$ **(A $\supset$ B)** 0100 $\wedge$ 0010 = **0000**
b. R16 **(A $\not\subset$ B)** $\vee$ **(A $\not\supset$ B)** 1011 $\vee$ 1101 = **1111**

Standard examples of De Morgan's Laws

(7) **0110 - 1001**

$\neg$[(A $\not\subset$ B) $\wedge$ (A $\not\supset$ B)] $\equiv$ (A $\subset$ B) $\vee$ (A $\supset$ B) (R9-R10)

$\neg$[(A $\subset$ B) $\vee$ (A $\supset$ B)] $\equiv$ (A $\not\subset$ B) $\wedge$ (A $\not\supset$ B)

1000 - 0001

$$\neg[(A \subseteq B) \wedge (A \supseteq B)] \equiv (A \not\subseteq B) \vee (A \not\supseteq B) \qquad \text{(R11a-R12a)}$$
$$\neg[(A \not\subseteq B) \vee (A \not\supseteq B)] \equiv (A \subseteq B) \wedge (A \supseteq B)$$
$$\neg[(A \not\subset B) \wedge (A \subseteq B)] \equiv (A \subset B) \vee (A \not\subseteq B) \qquad \text{(R11b-R12b)}$$
$$\neg[(A \subset B) \vee (A \not\subseteq B)] \equiv (A \not\subset B) \wedge (A \subseteq B)$$
$$\neg[(A \not\supset B) \wedge (A \supseteq B)] \equiv (A \supset B) \vee (A \not\supseteq B) \qquad \text{(R11c-R12c)}$$
$$\neg[(A \supset B) \vee (A \not\supseteq B)] \equiv (A \not\supset B) \wedge (A \supseteq B)$$

0001 - 1110

$$\neg[(A \not\subseteq B) \wedge (A \not\supseteq B)] \equiv (A \subseteq B) \vee (A \supseteq B) \qquad \text{(R13a-R14a)}$$
$$\neg[(A \subseteq B) \vee (A \supseteq B)] \equiv (A \not\subseteq B) \wedge (A \not\supseteq B)$$
$$\neg[(A \not\subset B) \wedge (A \not\supseteq B)] \equiv (A \subset B) \vee (A \supseteq B) \qquad \text{(R13b-R14b)}$$
$$\neg[(A \subset B) \vee (A \supseteq B)] \equiv (A \not\subset B) \wedge (A \not\supseteq B)$$
$$\neg[(A \not\subseteq B) \wedge (A \not\supset B)] \equiv (A \subseteq B) \vee (A \supset B) \qquad \text{(R13c-R14c)}$$
$$\neg[(A \subseteq B) \vee (A \supset B)] \equiv (A \not\subseteq B) \wedge (A \not\supset B)$$

0000 - 1111

$$\neg[(A \subset B) \wedge (A \supset B)] \equiv (A \not\subset B) \vee (A \not\supset B) \qquad \text{(R15-R16)}$$
$$\neg[(A \not\subset B) \vee (A \not\supset B)] \equiv (A \subset B) \wedge (A \supset B)$$

1.4 Set Inclusion in RDH and beyond

at this point one could straightforwardly:

=> “no surprise” :

8 relations with 4-bit definition => 8 relations are missing

=> Boolean closure

=> describe beta3/ THH or RDH for Aristotelian/Opposition/Implication geometries for Set Inclusion relations

=> derive Aristotelian Squares and Hexagons of Set Inclusion

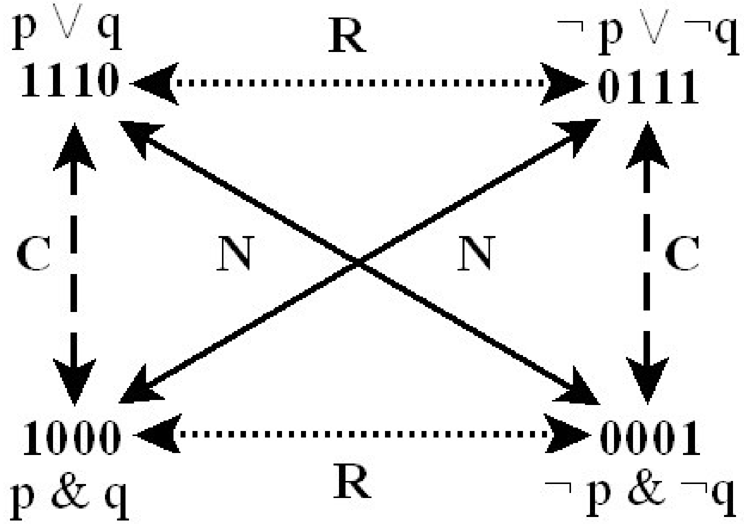
=> introduce the Extension from beta3 to beta4 using the 5-partition of the Gergonne relations instead of a 4-partition

2 Duality and reversibility in the Set Inclusion Algebra

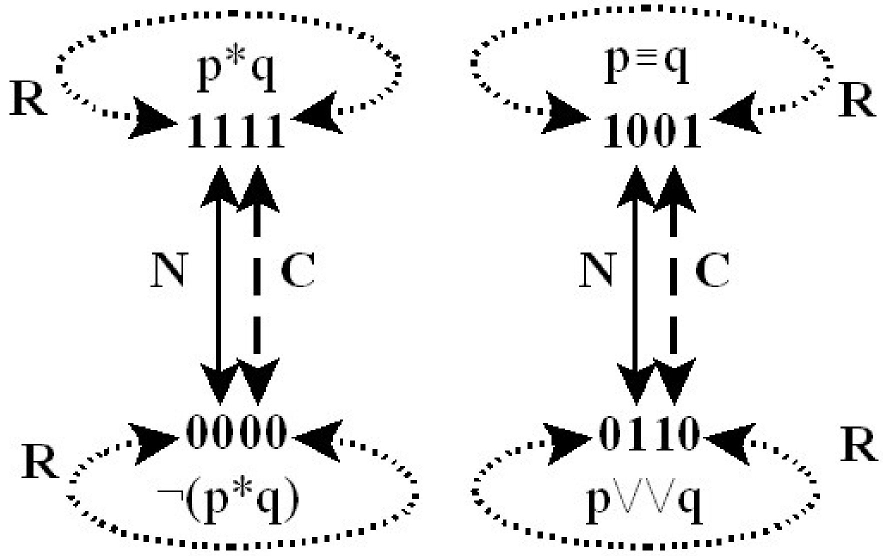
2.1 Duality in Piaget and Gottschalk

Duality	**Piaget (1949)**	**Gottschalk (1953)**	**Löbner (1990)**
corner	Identité (I)	Identity (E)	
Diagonal	Inversion (N)	Negational (N)	Negation (NEG)
Horizontal	Réciprocation (R)	Contradual (C)	Subnegation (SNEG)
Vertical	Corrélation (C)	Dual (D)	Dual (DUAL)

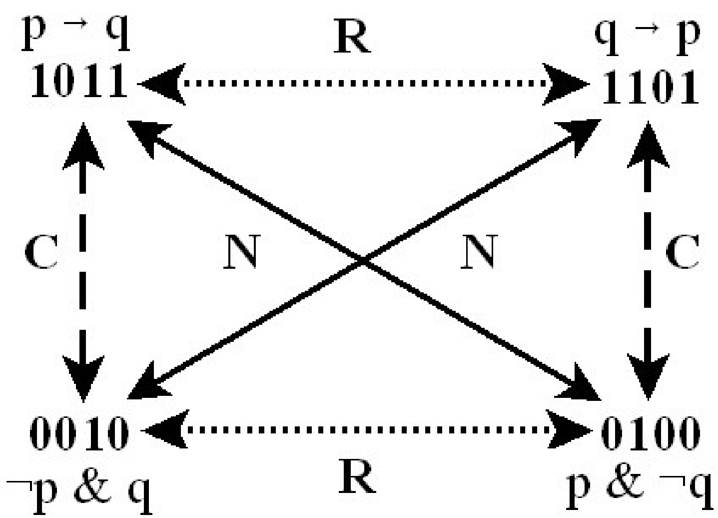
Piaget (1949)	**Gottschalk (1953)**	**(2012)**
I(abcd) = (abcd)		
N(abcd) = (a’b’c’d’)	Counterchange = interchange T/F	**switch**
R(abcd) = (dcba)	Invert = turn column upside down	**flip**
C(abcd) = (d’c’b’a’)	Transpose = invert + counterchange	**switch + flip**



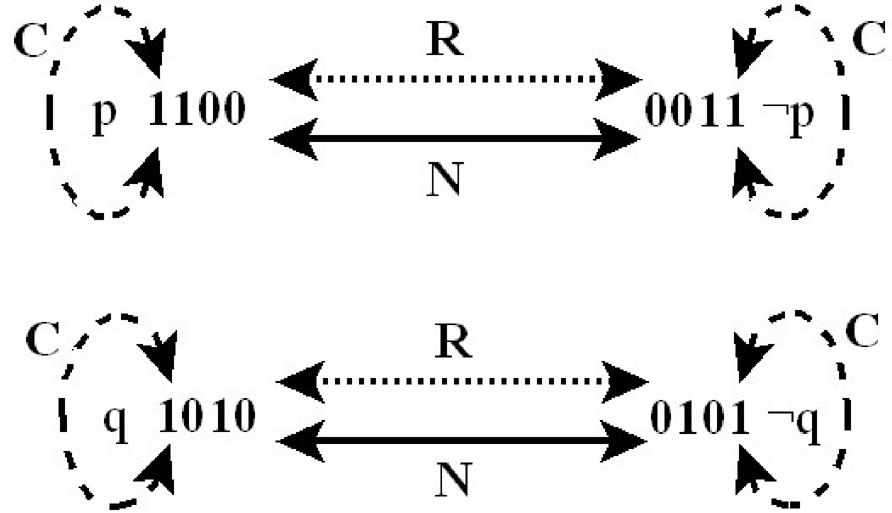
Quaternal A



Quaternal C



Quaternal B



Quaternal D

criticism of Blanché on the

=> Piaget analysis:

1. terminology “quaterns” C and D are in fact no real squares but two independent degenerate squares
2. orientation of quaterns is “upside down” from Aristotelian point of view: In Piaget’s quaterns the arrows go upwards

=> Gottschalk analysis (only quatern A is reversed upside down):

1. quatern A generated on basis of conjunction = L1 (1000) and quatern B generated on basis of implication = L3 (1011)
2. quatern A junctions : dual/C and negation/N are basic and subneg/R is derived (C+N=R)

   quatern B implications: subneg/R and negation/N are basic and dual/C is derived (R+N=C)

   => is key property of duality, not a problem!!

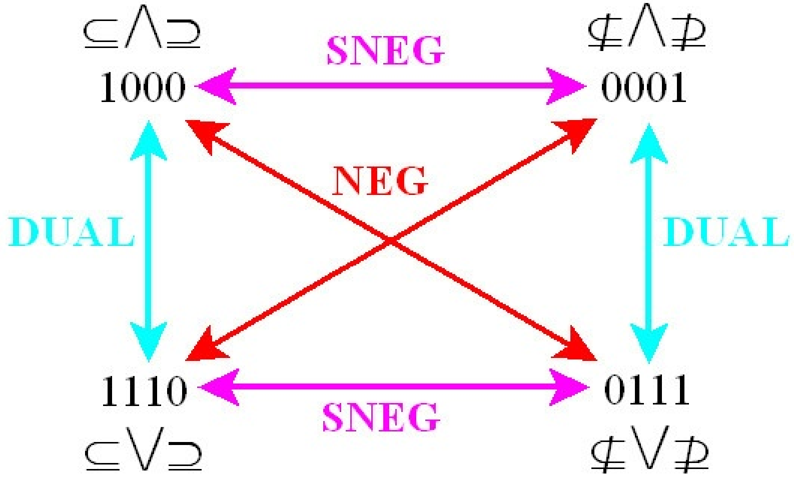
2.2 Duality and symmetry

distinguish 1. internal from external symmetry :
2. odd vs even levels of bitstrings L0-L4 ~ number of values 1

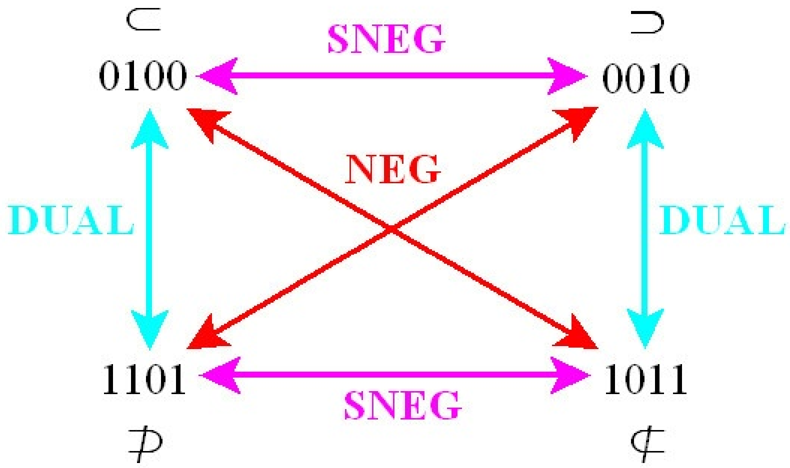
symmetry	*a=d*	*b=c*	*bit strings*	*arity*	*levels*
internal symmetry	-	+	0001 1000 1110 0111 quatern A of junctions	odd	L1+L3
external symmetry	+	-	0100 0010 1011 1101 quatern B of implications	odd	L1+L3
full symmetry	+	+	0000 1111 1001 0110 "quatern C" (R=I)	even	L0+L2+L4
no symmetry	-	-	1010 0101 1100 0011 "quatern D" (R=N)	even	L2

=> even levels = full or no symmetry
=> odd levels = external or internal symmetry

2.3 Duality geometry: squares and crosses



Quaternal A



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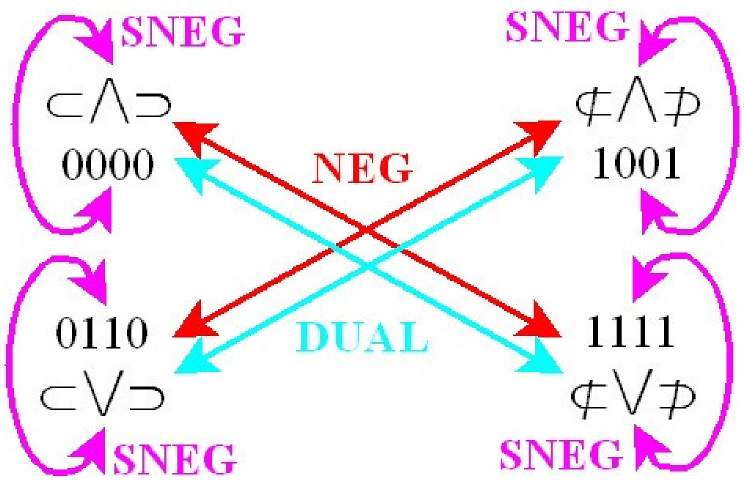
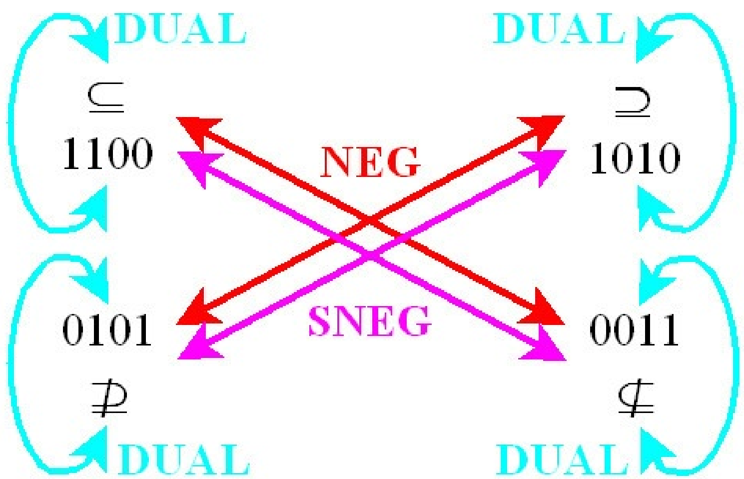
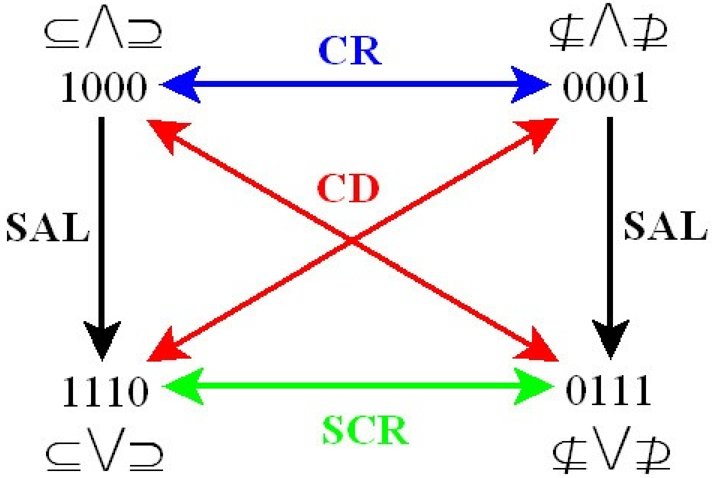


Quaternal C



Quaternal D

2.4 Aristotelian geometry: squares and crosses



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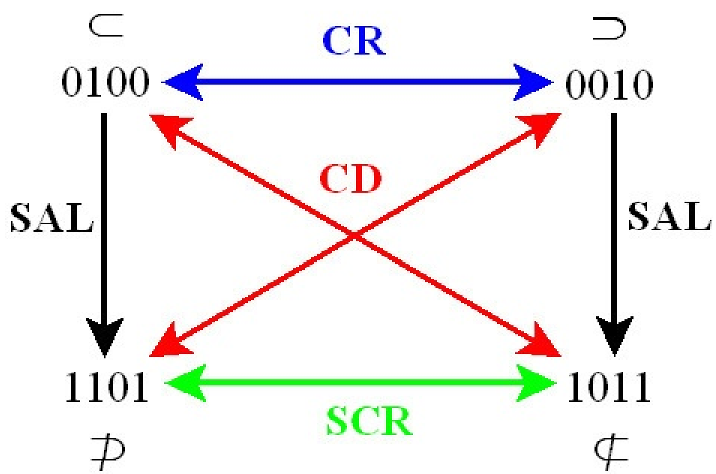

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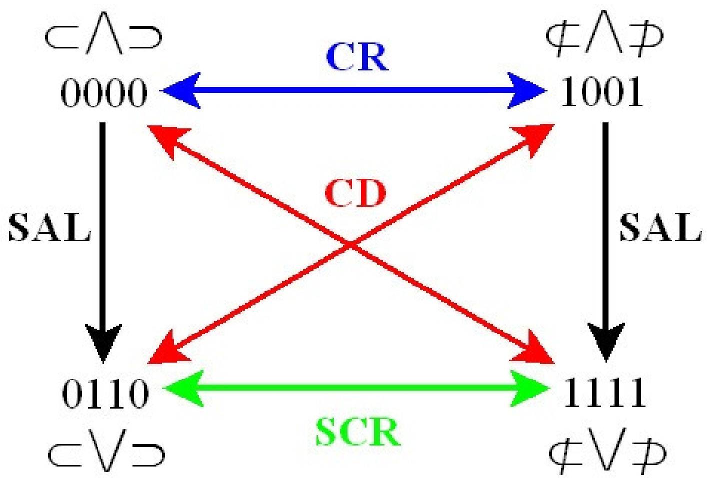
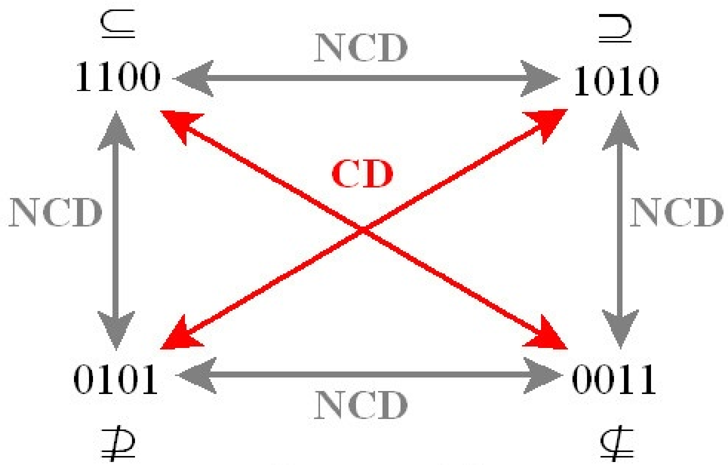


Quaternal C

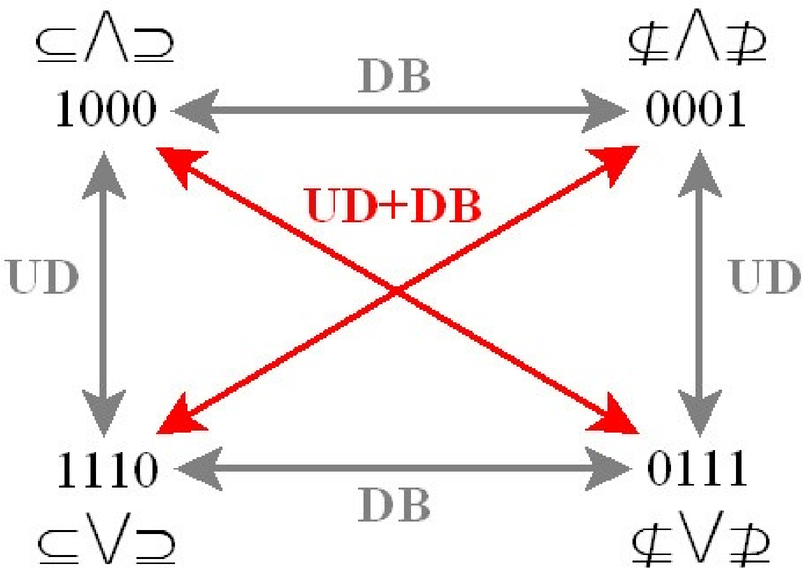


Quaternal D

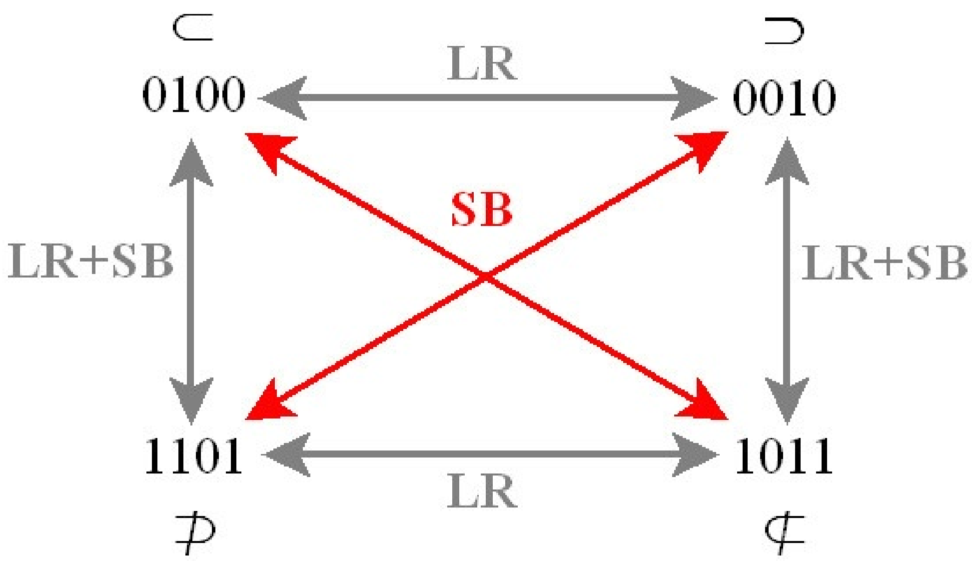
2.5 Reversibility geometry: squares

UD = up-down (horizontal mirror)
LR = left-right (vertical mirror)

SB = single bar for negation
DB = double bar for negation



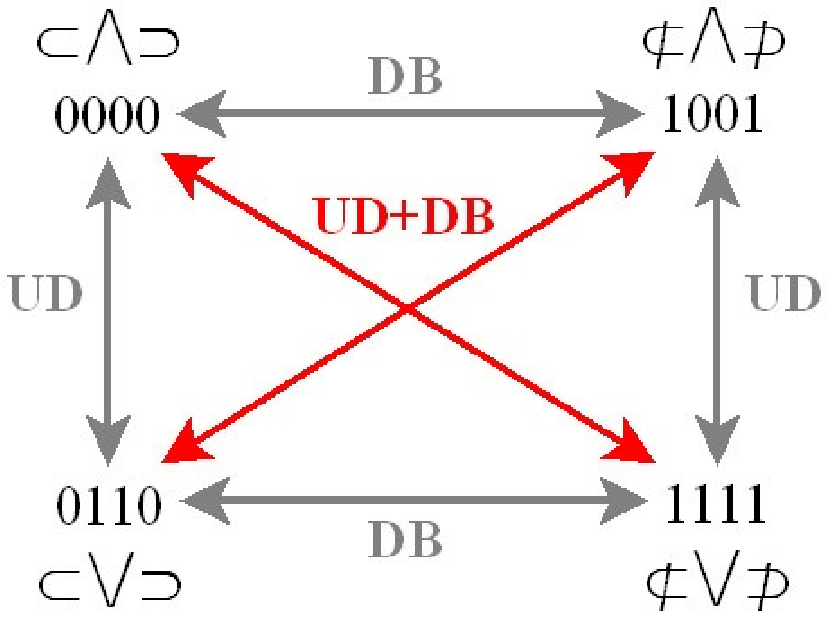
Quaternal A



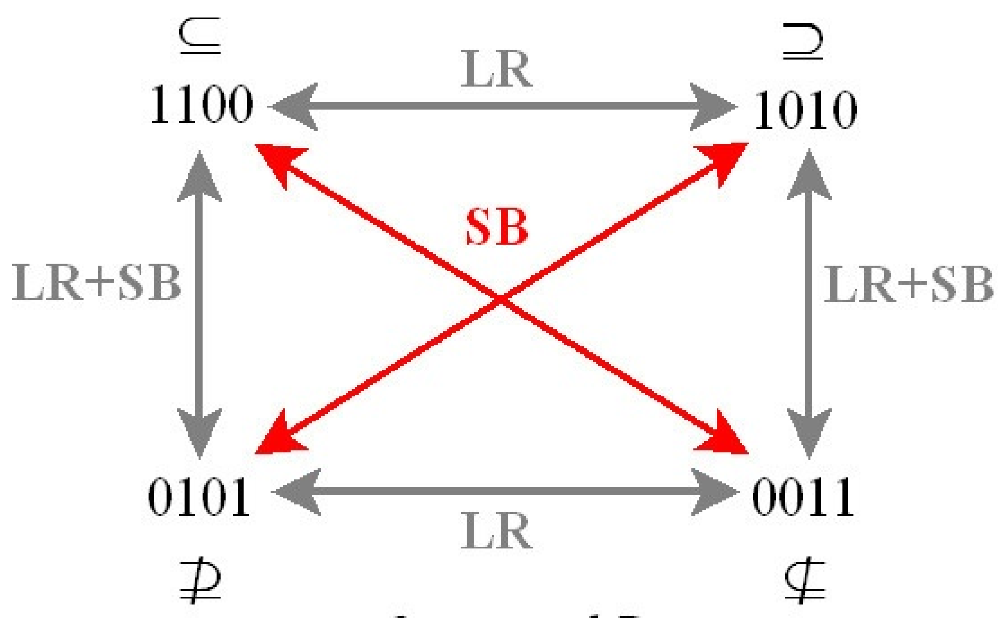
Quaternal B

Type1 reversibility (pred+arg)	**neg = sneg + dual**	“simple” dual
Type2 reversibility (pred+prop)	**dual = sneg + neg**	“complex” dual

(pace Blanché versus PG)



Quaternal C



Quaternal D

=> Quat C en D: same difference between reversibility type 1 type 2!!!

=> Reversibility Geometry (syntactic operations on formulae) is logically independent from the Duality Geometry (operations on bitstrings):

Duality = 2 squares + 2 crosses

Reversibility = 2 type 1 squares + 2 type 2 squares

=> NEXT STEP: decompose Duality Geometry

‘horizontal’ mirroring operations => **Flip Geometry**

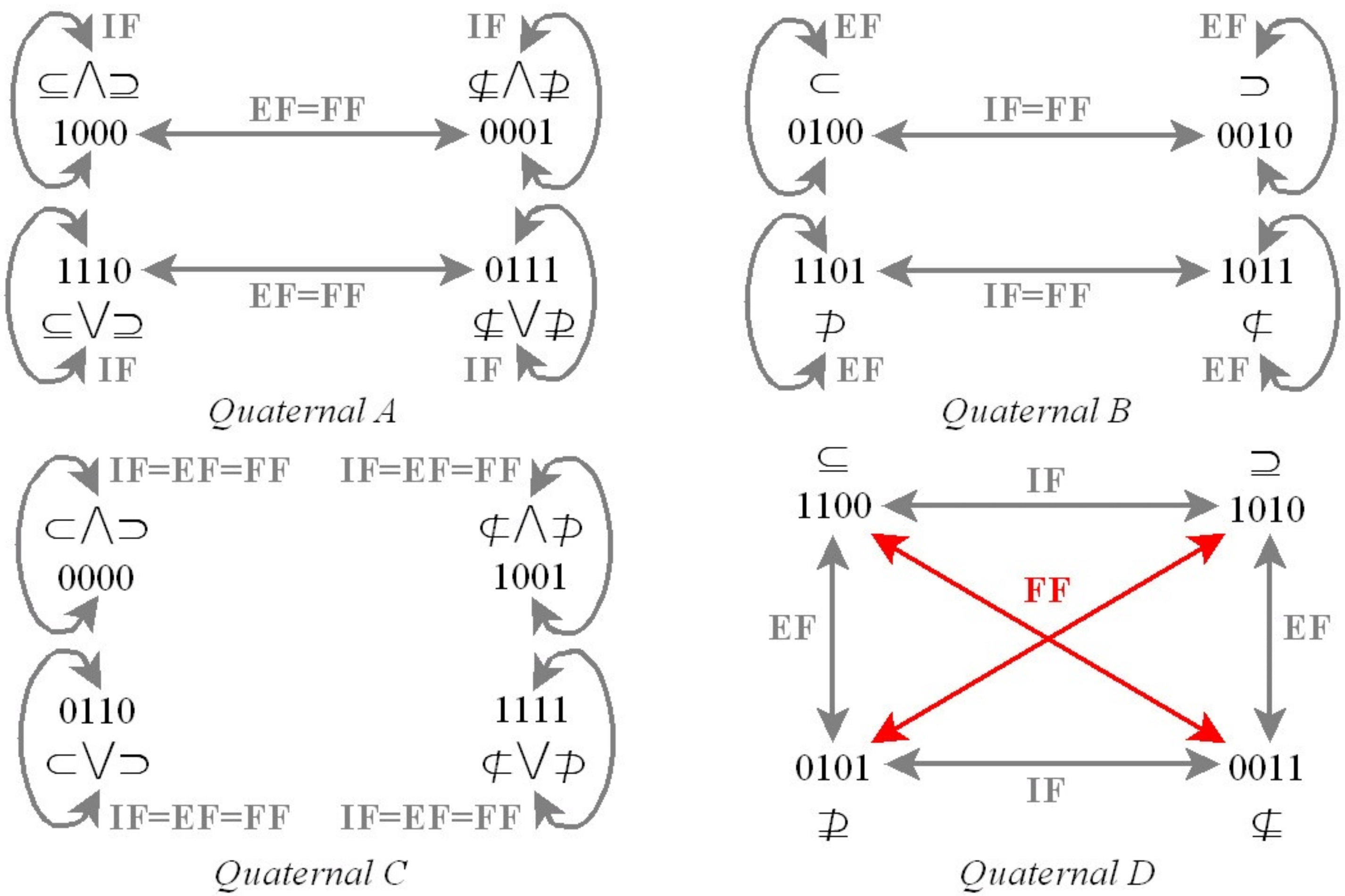
‘vertical’ polarity operations => **Switch Geometry**

2.6 Flip geometry: squares, bars and loops

IF = internal flip (abcd => acbd)

EF = external flip (abcd => dbca)

FF = full flip (abcd => dcba)



Quaternal A

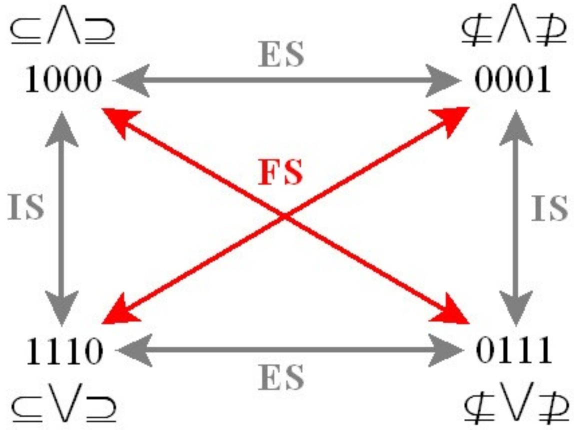
Quaternal B

Quaternal C

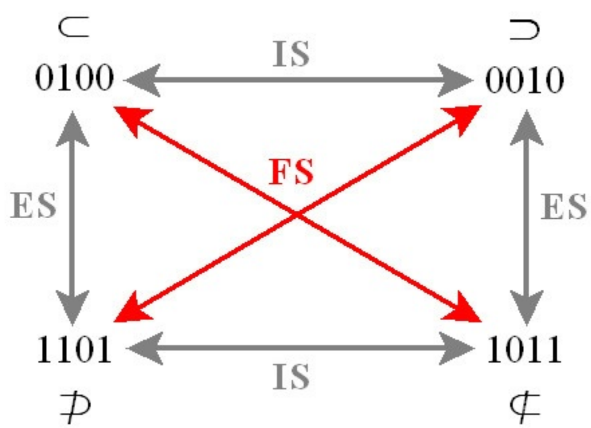
Quaternal D

2.7 Switch geometry: squares

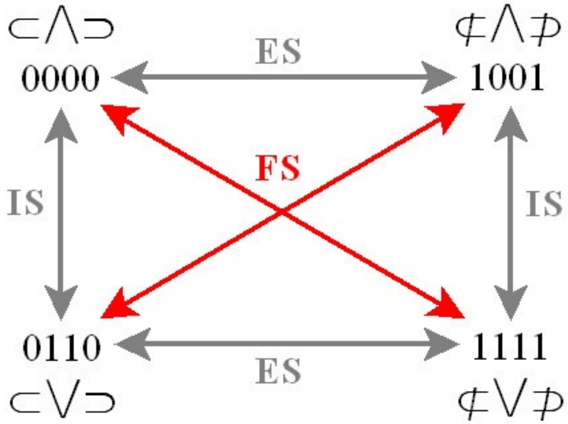
IS: internal switch (abcd => ab’c’d)   FS: full switch (abcd => a’b’c’d’)
ES: external switch (abcd => a’bcd’)



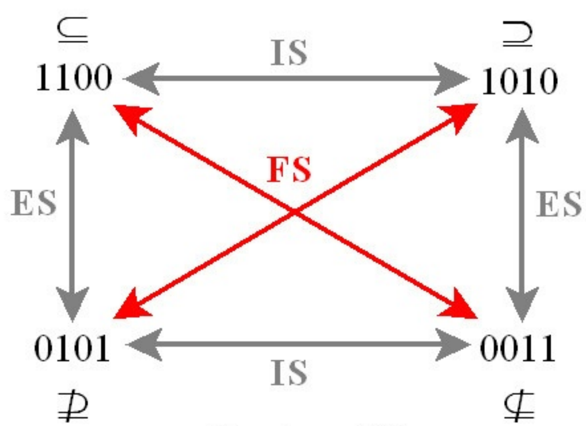
Quaternal A



Quaternal B



Quaternal C



Quaternal D

2.8 The hybrid nature of the Duality Geometry

=> Switch Geometry **<= Duality Geometry =>** Flip Geometry

Switch Geometry	Duality Geometry	Flip Geometry
4 squares	2 squares	1 square
internal switch	**full flip**	internal flip
external switch	**full switch**	external flip
full switch		**full flip**

=> Link with information perspective
~ hybrid nature of Aristotelian Geometry

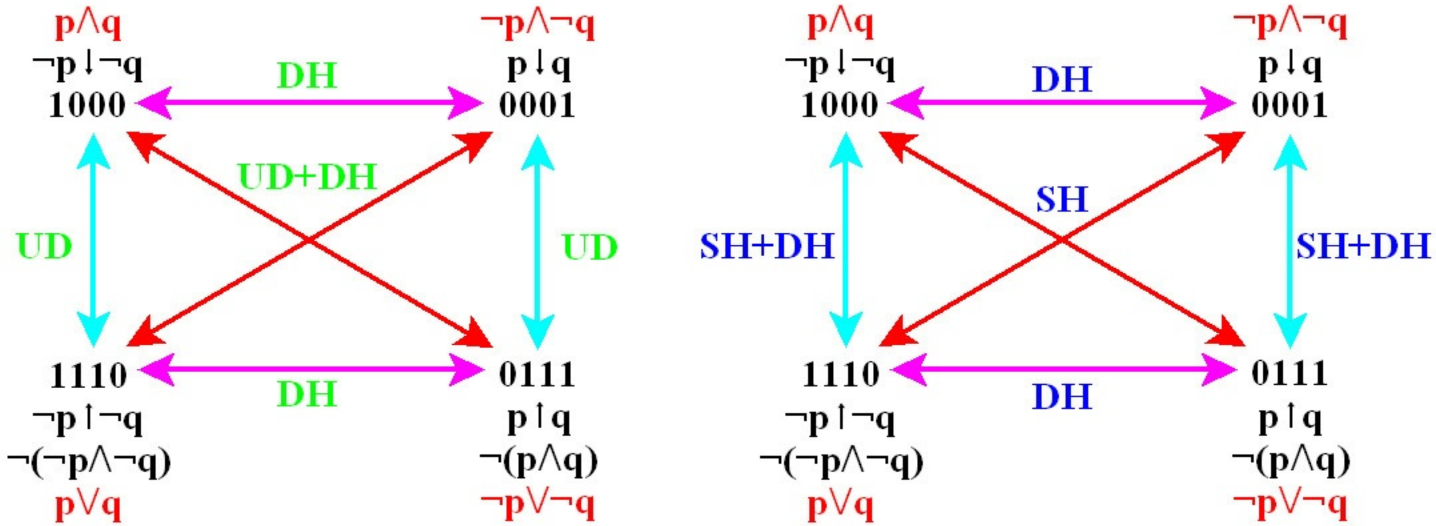
Switch Geom **>? Reversibility Geom >?** Duality Geom > Flip Geom

Switch Geom	Reversibility Geom	Duality Geom	Flip Geom
4 squares	4/3/2 squares	2 squares	1 square

2.9 Duality & reversibility with Propositional Connectives

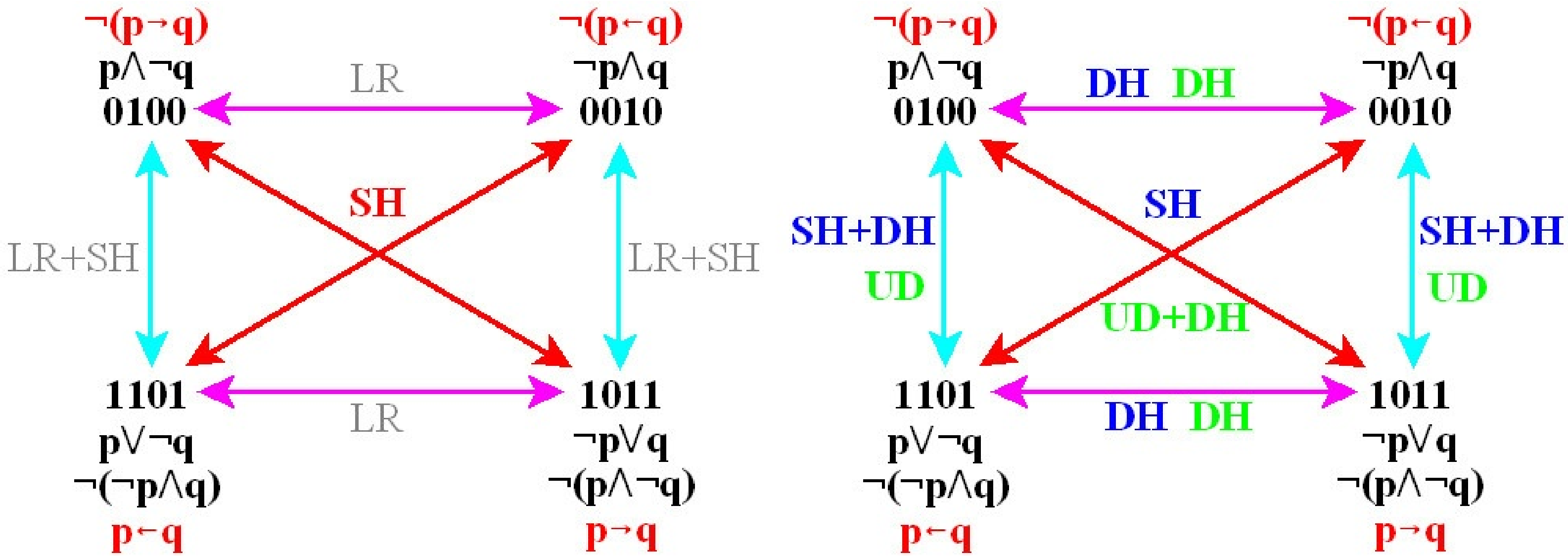
UD = up-down (horizontal mirror)
LR = left-right (vertical mirror)

SH = single hook for negation
DH = double hook for negation

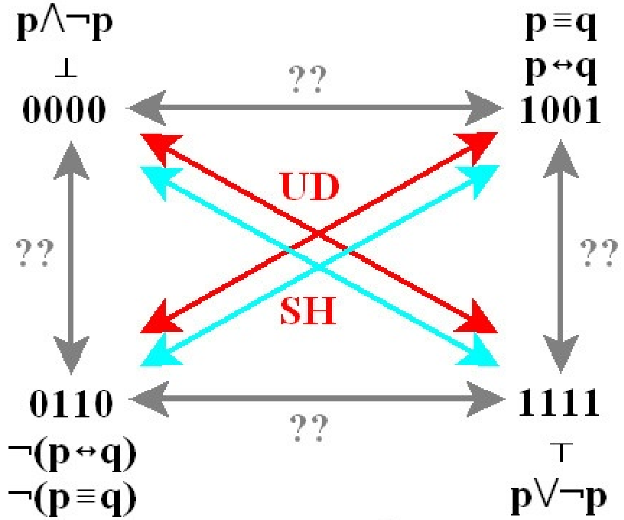


Quaternal A

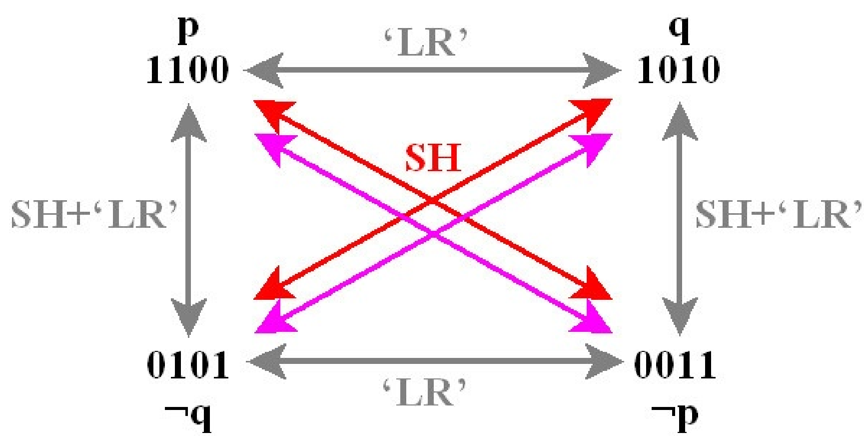
Type1 reversibility (pred+arg)	**neg = sneg + dual**	"simple" dual
Type2 reversibility (pred+prop)	**dual = sneg + neg**	"complex" dual
Type3 reversibility (arg+prop)	**dual = sneg + neg**	"complex" dual



Quaternal B



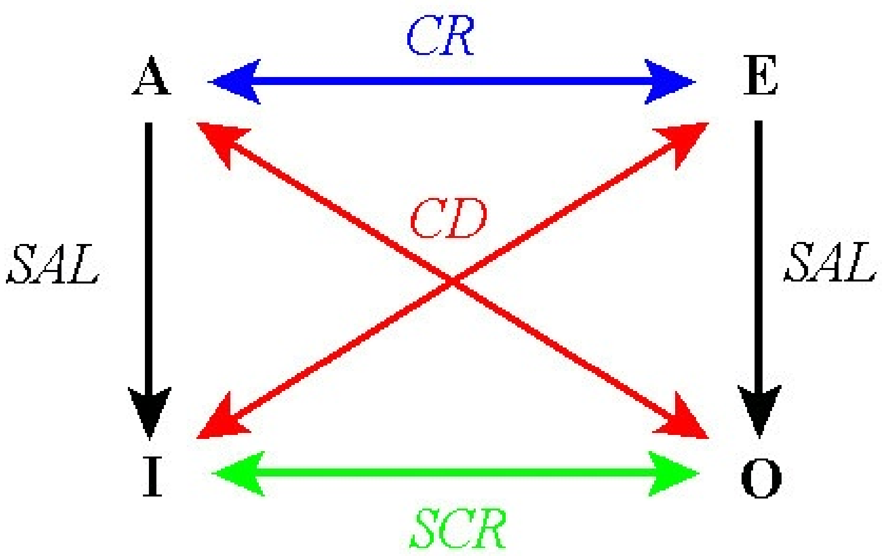
Quaternal C

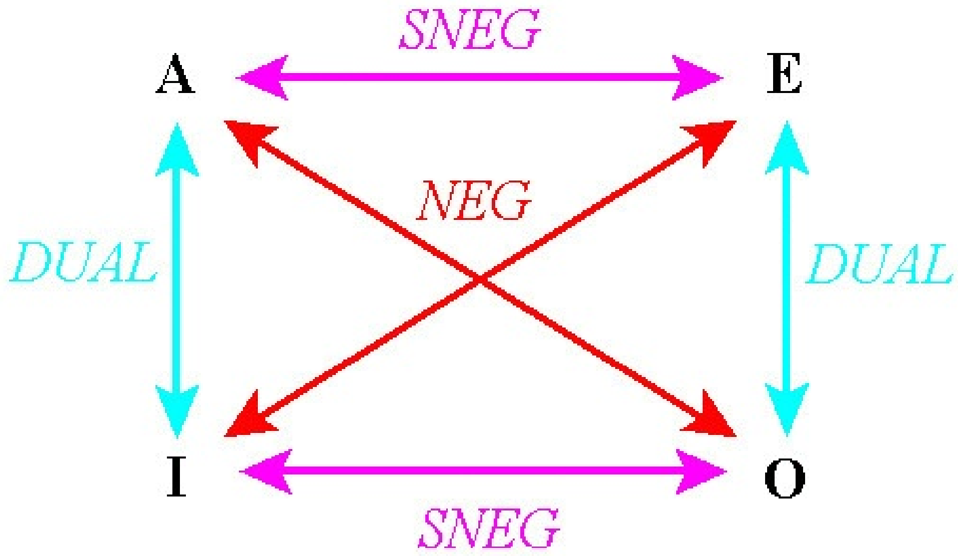


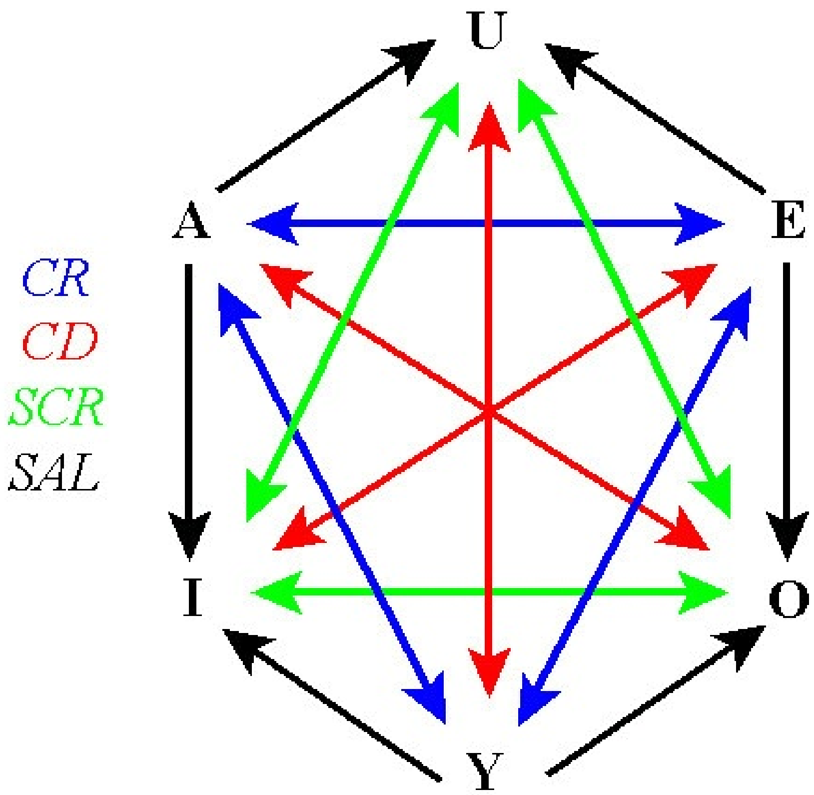
Quaternal D

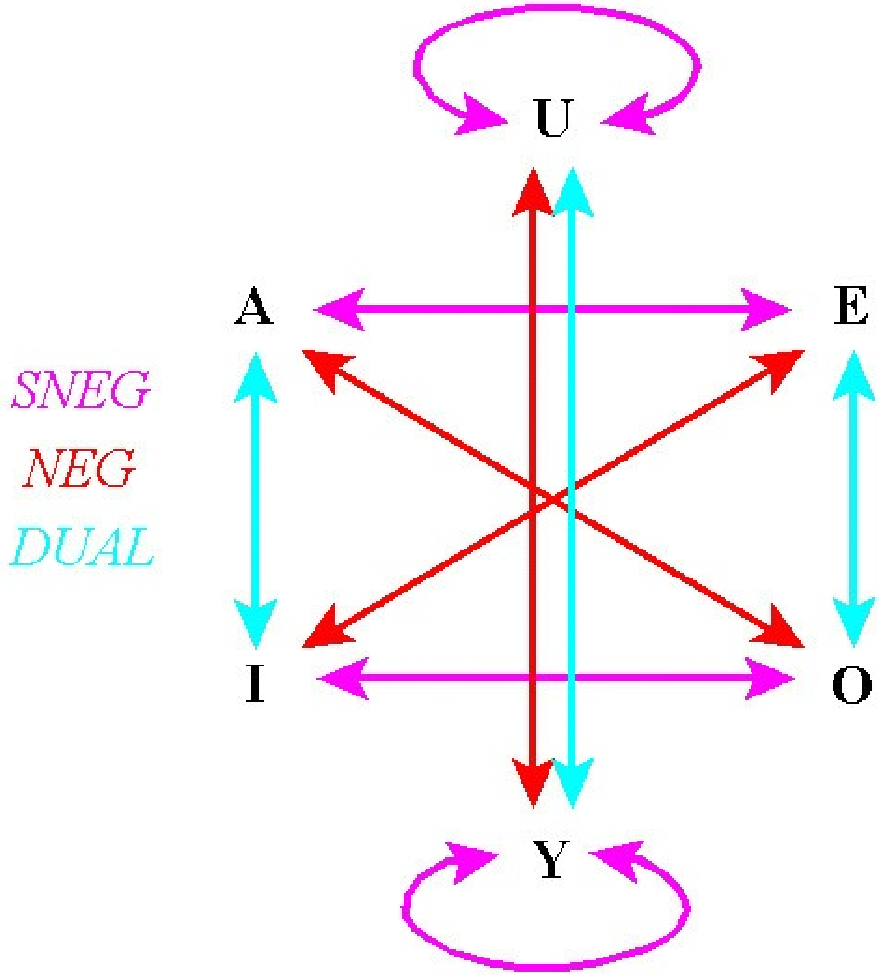
3 Duality squares and crosses from 2D to 3D

3.1 Duality vs Aristotelian relations in square/hexagon



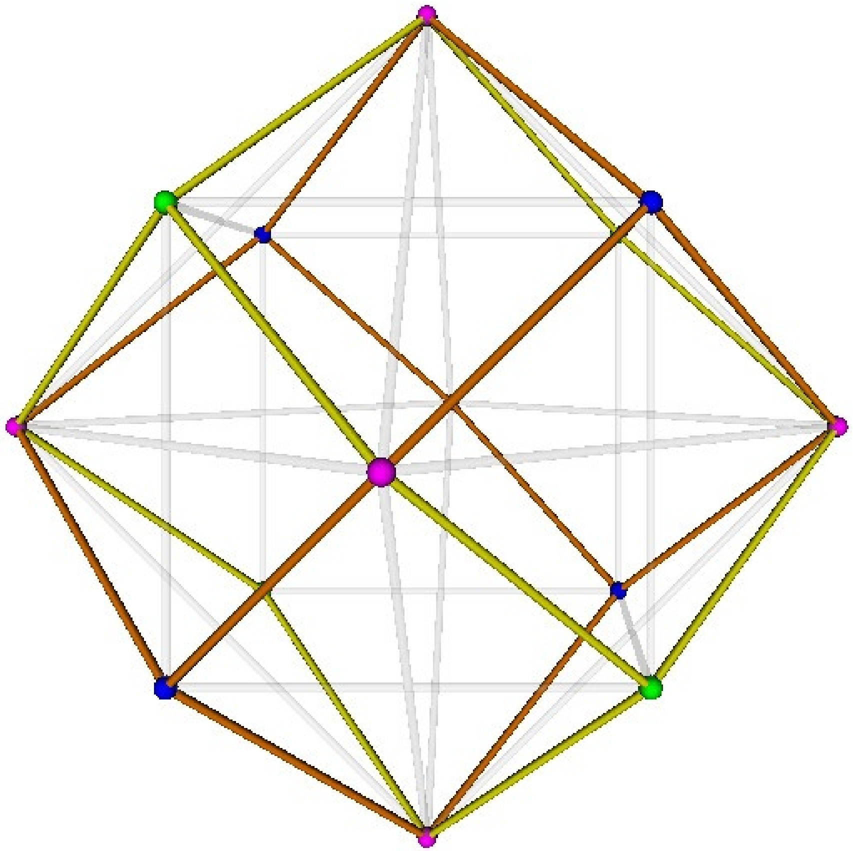


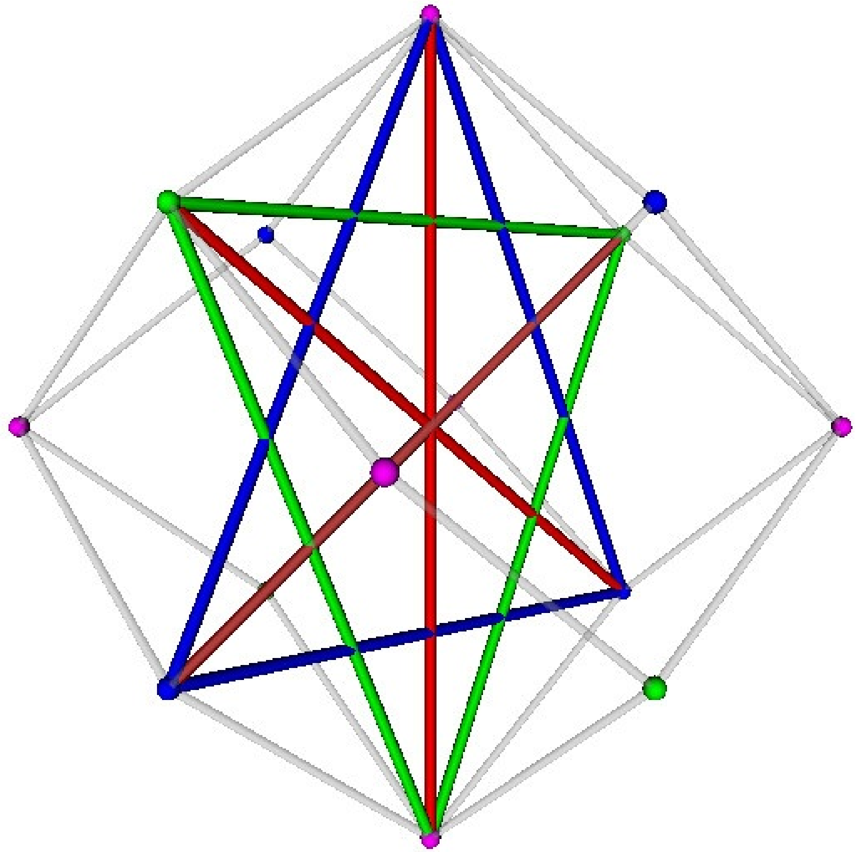

U
A
E
I
O
Y
CR
CD
SCR
SAL


U
A
E
I
O
Y
SNEG
NEG
DUAL

3.2 Aristotelian relations in RDH

"6 pairwise interlocking stars"





3.3 Duality relations in 2D: Piaget

Piaget (1952:147) *2D logical geometry:* mirror-operations in 4x4 table

Inverse (N) of an element X is its symmetric element w.r.t. the center of the square/table

$\neg(p*q)$; $\bot$ 0000	$\neg(p\rightarrow q)$; $(p\&\neg q)$ 0100	$(p\&q)$ 1000	p 1100
$\neg(q\rightarrow p)$;$(\neg p\&q)$ 0010	$p\vee\vee q$; $\neg(p\leftrightarrow q)$ 0110	q 1010	$p\vee q$ 1110
$\neg(p\vee q)$; $(\neg p\&\neg q)$ 0001	$\neg q$ 0101	$p\equiv q$; $p\leftrightarrow q$ 1001	$q\rightarrow p$; $\neg(\neg p\&q)$ 1101
$\neg p$ 0011	$p\vert q$ $\neg(p\&q)$ 0111	$p\rightarrow q$; $\neg(p\&\neg q)$ 1011	$p*q$; $\top$ 1111

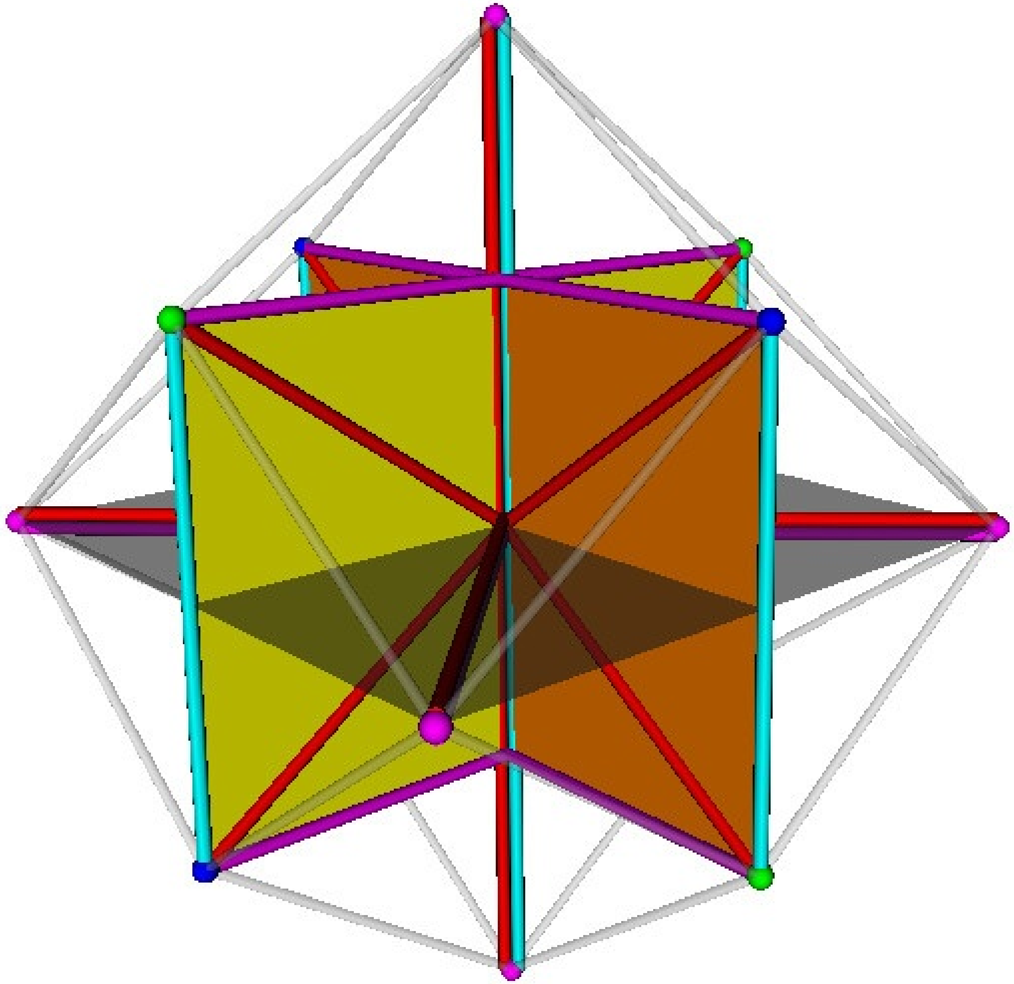
Réciproque (R) is symmetric element w.r.t. "decreasing" diagonal

[illegible]); ⊥ 000[illegible]	¬(p→q); (p&¬q) 0100	(p&q) 1000	p 1100
¬(q→p) ;(¬p&q) 0010	p[illegible](p↔q) 011[illegible]	q 1010	p∨q 1110
¬(p∨q); (¬p&¬q) 0001	¬q 0101	[illegible]↔q 100[illegible]	q→p ; ¬(¬p&q) 1101
¬p 0011	p|q ¬(p&q) 0111	p→q ; ¬(p&¬q) 1011	[illegible] ⊤ 111[illegible]

Corrélative (C) is symmetric element w.r.t. the "increasing" diagonal

¬(p*q); ⊥ 0000	¬(p→q); (p&¬q) 0100	(p&q) 1000	[illegible]100
¬(q→p) ;(¬p&q) 0010	p∨∨q; ¬(p↔q) 0110	[illegible]010	p∨q 1110
¬(p∨q); (¬p&¬q) 0001	[illegible]101	p≡q; p↔q 1001	q→p ; ¬(¬p&q) 1101
[illegible]011	p|q ¬(p&q) 0111	p→q ; ¬(p&¬q) 1011	p*q ; ⊤ 1111

3.4 Duality relations in 3D: RDH



4 Conclusion and prospects

=> Algebra of Set-Inclusion as β3/TTH/RDH

=> Distinguish duality squares from duality crosses

=> Decompose Duality geometry in two steps:

1. distinction between
   syntax: operations on formulae => **Reversibility Geometry**
   semantics: operations on bitstrings
2. within operations on bitstrings distinction between
   ‘horizontal’ mirroring operations => **Flip Geometry**
   ‘vertical’ polarity operations => **Switch Geometry**

=> Switch Geometry **<= Duality Geometry =>** Flip Geometry

Switch Geometry	Duality Geometry	Flip Geometry
4 squares	2 squares	1 square

Switch Geom **>? Reversibility Geom >?** Duality Geom > Flip Geom

Switch Geom	Reversibility Geom	Duality Geom	Flip Geom
4 squares	4/3/2 squares	2 squares	1 square

downgrade analysis of duality from 3D/4-bit in RDH
to 2D/3-bit in hexagon

quaterns		external symmetry	
		-	+
internal symmetry	-	1010 0101 1100 0011 “quatern D” (R=N)	0100 0010 1011 1101 quatern B of implications
	+	0**00**1 1**00**0 1**11**0 0**11**1 quatern A of junctions	0**00**0 1**11**1 1**00**1 0**11**0 “quatern C” (R=I)

quaterns		external symmetry	
		-	+
internal symmetry	-		
	+	0**0**1 1**0**0 1**1**0 0**1**1 duality square	0**0**0 1**1**1 1**0**1 0**1**0 duality cross (R=I)

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