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Aristotelian diagrams for the proportional quantifier ‘most’

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Square 2018

- **Aim**: study the **interaction** between 2 **four-formula fragments** that independently yield an opposition diagram:
  - the fragment $\mathcal{F}_{FOL}$ with the **four FOL quantifiers**:

$$\mathcal{F}_{FOL} := \{ \begin{array}{ll} \text{all(A,B)}, & |A \setminus B| = 0 \\ \text{no(A,B)}, & |A \cap B| = 0 \\ \text{some(A,B)}, & |A \cap B| > 0 \\ \text{not all(A,B)} \} & |A \setminus B| > 0 \end{array}$$

  - the fragment $\mathcal{F}_{most}$ generated on the basis of the **proportional quantifier** *most*. In Generalized Quantifier Theory (GQT): *most*(A,B) ≈ *more than half*(A,B):

$$\mathcal{F}_{most} := \{ \begin{array}{ll} \text{most(A,B)}, & |A \cap B| > |A \setminus B| \\ \text{most(A,}\neg\text{B)}, & |A \cap B| < |A \setminus B| \\ \neg\text{most(A,}\neg\text{B)}, & |A \cap B| \geq |A \setminus B| \\ \neg\text{most(A,B)} \} & |A \cap B| \leq |A \setminus B| \end{array}$$

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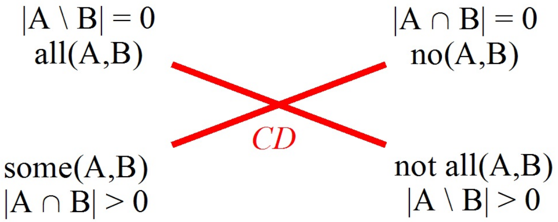
Square for 'all'



- universal quantifiers *all* and *no* lack existential import in FOL
- the resulting constellation is a so-called **degenerate square**
  - only two *contradiction* (CD) relations on the diagonals
  - no *contrariety* (C) between *all* and *no*
  - no *subcontrariety* (SC) between *some* and *not all*
  - no *subalternation* (SA) from *all* to *some*, nor from *no* to *not all*
  - four pairs of *unconnectedness* (no Aristotelian relation whatsoever)
  - 'X of opposition'

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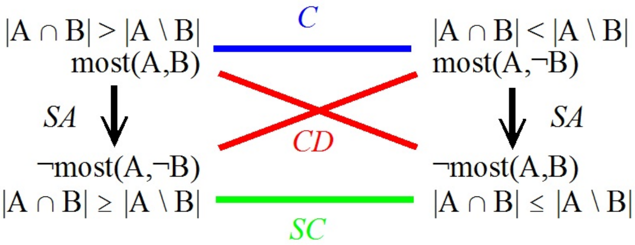

- proportional quantifier *most* does have existential import: if *most A are B* then there is at least one A: $|A \cap B| > |A \setminus B|$ entails $|A| > 0$.
- the resulting constellation is a so-called **classical Aristotelian square**
  - two *contradiction* (CD) relations on the diagonals
  - two *subalternation* (SA) relations
  - one contrariety (C) and one subcontrariety (SC) relation

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2. Aristotelian squares
   - Squares for ‘all’ and ‘most’
   - **Bitstring semantics for ‘all’ and ‘most’**
3. Aristotelian octagons
   - A first octagon for ‘all’ + ‘most’
   - Bitstring semantics for ‘all’ + ‘most’
   - A second octagon for ‘all’ + ‘most’
4. Conclusion

- The fragment $\mathcal{F}_{FOL}$ induces the **partition** $\Pi(\mathcal{F}_{FOL}) =$

$$\begin{array}{llr}
\{ & |A \setminus B| = 0 \;\; \& \;\; |A| > 0, & \alpha_1\text{: all A are B \& there are A's} \\
 & |A \setminus B| > 0 \;\; \& \;\; |A \cap B| > 0, & \alpha_2\text{: some but not all A are B} \\
 & |A \cap B| = 0 \;\; \& \;\; |A| > 0, & \alpha_3\text{: no A are B \& there are A's} \\
 & |A| = 0 \qquad\qquad \} & \alpha_4\text{: there are no A's}
\end{array}$$

$\Rightarrow$ quadripartition of logical space using 4 anchor formulas

- The **bitstring semantics** β_{FOL} for the fragment $\mathcal{F}_{FOL}$:

$$\begin{array}{lclr}
\beta_{FOL}(all(A,B)) & = & \mathbf{1001} & |A \setminus B| = 0 \\
\beta_{FOL}(no(A,B)) & = & \mathbf{0011} & |A \cap B| = 0 \\
\beta_{FOL}(some(A,B)) & = & \mathbf{1100} & |A \cap B| > 0 \\
\beta_{FOL}(not\ all(A,B)) & = & \mathbf{0110} & |A \setminus B| > 0
\end{array}$$

$\Rightarrow$ a degenerate square requires bitstrings of length 4 (Demey & Smessaert, 2018)

- The fragment $\mathcal{F}_{most}$ induces the **partition** $\Pi(\mathcal{F}_{most}) =$

$$\begin{array}{lll} \{ & |A \cap B| > |A \setminus B|, & \alpha'_1\text{: More than half (A,B)} \\ & |A \cap B| = |A \setminus B|, & \alpha'_2\text{: Exactly half (A,B)} \\ & |A \cap B| < |A \setminus B| \quad \} & \alpha'_3\text{: Less than half (A,B)} \end{array}$$

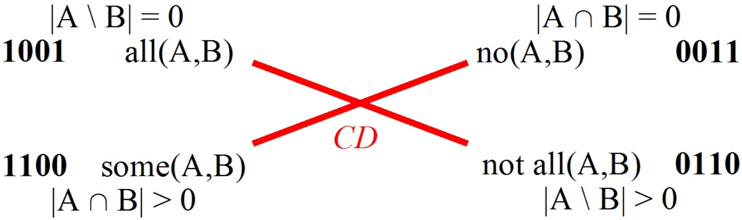
$\Rightarrow$ tripartition of logical space using 3 anchor formulas

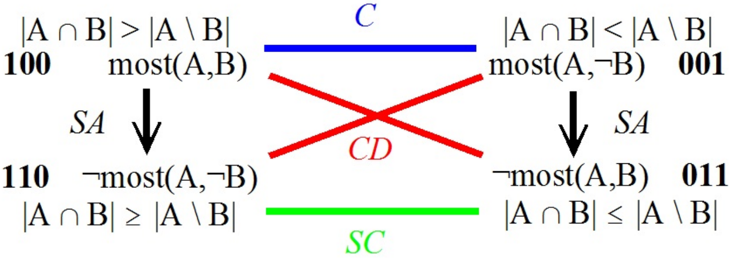
- The **bitstring semantics** β_{most} for the fragment $\mathcal{F}_{most}$:

$$\begin{array}{lcll} \beta_{most}(most(A,B)) & = & \mathbf{100} & |A \cap B| > |A \setminus B| \\ \beta_{most}(most(A,\neg B)) & = & \mathbf{001} & |A \cap B| < |A \setminus B| \\ \beta_{most}(\neg most(A,\neg B)) & = & \mathbf{110} & |A \cap B| \geq |A \setminus B| \\ \beta_{most}(\neg most(A,B)) & = & \mathbf{011} & |A \cap B| \leq |A \setminus B| \end{array}$$

$\Rightarrow$ a classical square only requires bitstrings of length 3.

Squares for ‘all’ and ‘most’





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- Combine the two four-formula fragments into one **eight-formula fragment** $\mathcal{F}_{FOLmost} := \mathcal{F}_{FOL} \cup \mathcal{F}_{most}$

$\mathcal{F}_{FOLmost}$:= {	all(A,B),		$\|A \setminus B\| = 0$
	no(A,B),		$\|A \cap B\| = 0$
	most(A,B),		$\|A \cap B\| > \|A \setminus B\|$
	most(A,¬B),		$\|A \cap B\| < \|A \setminus B\|$
	¬most(A,¬B),		$\|A \cap B\| \geq \|A \setminus B\|$
	¬most(A,B),		$\|A \cap B\| \leq \|A \setminus B\|$
	some(A,B),		$\|A \cap B\| > 0$
	not all(A,B)	}	$\|A \setminus B\| > 0$

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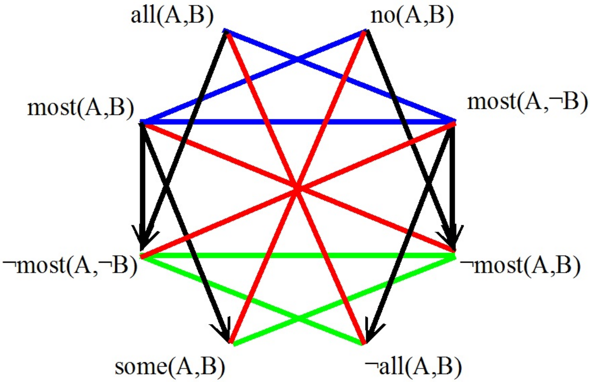
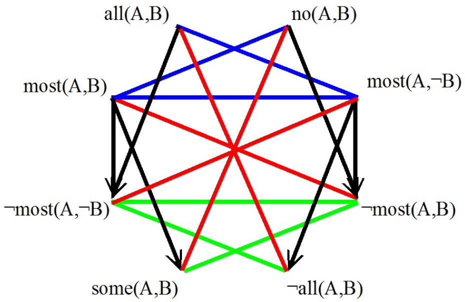


- Interlock two squares into an octagon; two subalternations are crucial:
  - from *all(A,B)* to *¬most(A,¬B)* $(|A \setminus B| = 0$ entails $|A \cap B| \geq |A \setminus B|)$
  - from *most(A,B)* to *some(A,B)* $(|A \cap B| > |A \setminus B|$ entails $|A \cap B| > 0)$



- octagon containing 3 classical squares and 3 degenerate squares:
  - known in theory as one of the 18 families of Aristotelian octagons.
  - Now first ‘non-artificial’ instantiation of that family
  - nicely fits into a series of octagons in which the number of degenerate squares increases from **zero** (Moretti 2009), over **one** (Buridan/Klima 2001), and **two** (Keynes 1884, Johnson 1921), to **three**.

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   - **Bitstring semantics for ‘all’ + ‘most’**
   - A second octagon for ‘all’ + ‘most’
4. Conclusion

$$\Pi(\mathcal{F}_{FOL}) = \{ \alpha_1: |A \setminus B| = 0 \ \& \ |A| > 0, \quad \alpha_2: |A \setminus B| > 0 \ \& \ |A \cap B| > 0, \quad \alpha_3: |A \cap B| = 0 \ \& \ |A| > 0, \quad \alpha_4: |A| = 0 \}$$

$$\Pi(\mathcal{F}_{most}) = \{ \alpha'_1: |A \cap B| > |A \setminus B|, \quad \alpha'_2: |A \cap B| = |A \setminus B|, \quad \alpha'_3: |A \cap B| < |A \setminus B| \}$$

- compute bitstring semantics by taking the meet of the two original partitions: $\Pi(\mathcal{F}_{FOLmost}) := \Pi(\mathcal{F}_{FOL}) \wedge_{FOL} \Pi(\mathcal{F}_{most})$
- compute the conjunctions of each anchor formula α_i of the former partition with each anchor formula α'_j of the latter.
- this yields $4 \times 3 = 12$ conjunctions.
- after elimination of the inconsistent formulas we get a hexapartition.

$$\Pi(\mathcal{F}_{FOL}) = \{ \alpha_1: |A \setminus B| = 0 \ \ \& \ \ |A| > 0, \quad \alpha_2: |A \setminus B| > 0 \ \ \& \ \ |A \cap B| > 0, \quad \alpha_3: |A \cap B| = 0 \ \ \& \ \ |A| > 0, \quad \alpha_4: |A| = 0 \}$$

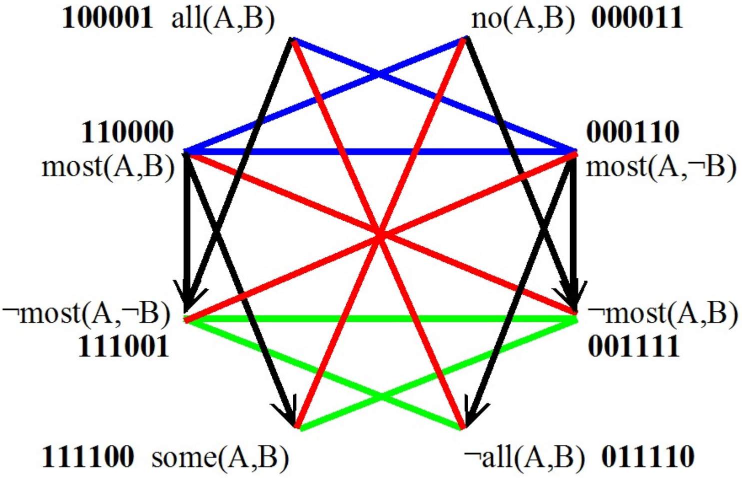
$$\Pi(\mathcal{F}_{most}) = \{ \alpha'_1: |A \cap B| > |A \setminus B|, \quad \alpha'_2: |A \cap B| = |A \setminus B|, \quad \alpha'_3: |A \cap B| < |A \setminus B| \}$$

- $\Pi(\mathcal{F}_{FOLmost}) := \Pi(\mathcal{F}_{FOL}) \wedge_{FOL} \Pi(\mathcal{F}_{most})$
- The fragment $\mathcal{F}_{FOLmost}$ induces the **partition** $\Pi(\mathcal{F}_{FOLmost}) =$

$\{ \ |A \cap B| > |A \setminus B| = 0,$	α''_1: All A are B and there are A's
$|A \cap B| > |A \setminus B| > 0,$	α''_2: Most but not all A's are B
$|A \cap B| = |A \setminus B| > 0,$	α''_3: Exactly half the A's are B
$0 < |A \cap B| < |A \setminus B|,$	α''_4: Most but not all A's are not B
$0 = |A \cap B| < |A \setminus B|,$	α''_5: No A's are B, but there are A's
$0 = |A \cap B| = |A \setminus B| \ \}$	α''_6: There are no A's

- The **bitstring semantics** $\beta_{FOLmost}$ for the fragment $\mathcal{F}_{FOLmost}$:

$$
\begin{array}{lcll}
\beta_{FOLmost}(all(A,B)) & = & \mathbf{100001} & |A \setminus B| = 0 \\
\beta_{FOLmost}(no(A,B)) & = & \mathbf{000011} & |A \cap B| = 0 \\
\beta_{FOLmost}(most(A,B)) & = & \mathbf{110000} & |A \cap B| > |A \setminus B| \\
\beta_{FOLmost}(most(A,\neg B)) & = & \mathbf{000110} & |A \cap B| < |A \setminus B| \\
\beta_{FOLmost}(\neg most(A,\neg B)) & = & \mathbf{111001} & |A \cap B| \geq |A \setminus B| \\
\beta_{FOLmost}(\neg most(A,B)) & = & \mathbf{001111} & |A \cap B| \leq |A \setminus B| \\
\beta_{FOLmost}(some(A,B)) & = & \mathbf{111100} & |A \cap B| > 0 \\
\beta_{FOLmost}(not\ all(A,B)) & = & \mathbf{011110} & |A \setminus B| > 0
\end{array}
$$


100001 all(A,B)
no(A,B) 000011
110000
most(A,B)
000110
most(A,¬B)
¬most(A,¬B)
111001
¬most(A,B)
001111
111100 some(A,B)
¬all(A,B) 011110

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4. Conclusion

- the fragment $\mathcal{F}_{SYLL}$ with the **four SYLL quantifiers**:

$$\mathcal{F}_{SYLL} \quad := \quad \{ \quad \begin{array}{rl} \text{all(A,B)}, & |A \setminus B| = 0 \ \ \& \ \ |A| > 0 \\ \text{no(A,B)}, & |A \cap B| = 0 \ \ \& \ \ |A| > 0 \\ \text{some(A,B)}, & |A \cap B| > 0 \\ \text{not all(A,B)} \quad \} & |A \setminus B| > 0 \end{array}$$

$\Rightarrow$ move from FOL to SYLL: 'all' and 'no' **do** have existential import

- The fragment $\mathcal{F}_{SYLL}$ induces the **partition** $\Pi(\mathcal{F}_{SYLL}) =$

$$\{ \quad \begin{array}{ll} |A \setminus B| = 0 \ \ \& \ \ |A| > 0, & \alpha_1\text{: all A are B \& there are A's} \\ |A \setminus B| > 0 \ \ \& \ \ |A \cap B| > 0, & \alpha_2\text{: some but not all A are B} \\ |A \cap B| = 0 \ \ \& \ \ |A| > 0 \quad \} & \alpha_3\text{: no A are B \& there are A's} \end{array}$$

$\Rightarrow$ anchor formulas α_1, α_2 and α_3 are the same as in $\Pi(\mathcal{F}_{FOL})$
α_4 is deleted because it has become inconsistent in SYLL.

- The **bitstring semantics** β_{SYLL} for the fragment $\mathcal{F}_{SYLL}$:

$$
\begin{array}{lcll}
\beta_{SYLL}(all(A,B)) & = & \mathbf{100} & |A \setminus B| = 0 \ \ \& \ \ |A| > 0 \\
\beta_{SYLL}(no(A,B)) & = & \mathbf{001} & |A \cap B| = 0 \ \ \& \ \ |A| > 0 \\
\beta_{SYLL}(some(A,B)) & = & \mathbf{110} & |A \cap B| > 0 \\
\beta_{SYLL}(not\ all(A,B)) & = & \mathbf{011} & |A \setminus B| > 0
\end{array}
$$

- Compared to $\Pi(\mathcal{F}_{FOL})$, anchor formula α_4 is deleted from $\Pi(\mathcal{F}_{SYLL})$
- so bit position four is deleted as well, yielding a bitstring semantics with bitstrings of length **three** instead of four.

A second octagon for 'all' + 'most'

- Combine the two four-formula fragments into one **eight-formula fragment** $\mathcal{F}_{SYLLmost} := \mathcal{F}_{SYLL} \cup \mathcal{F}_{most}$

$$
\mathcal{F}_{SYLLmost} := \{ \begin{array}{rl}
\text{all(A,B)}, & |A \setminus B| = 0 \;\; \& \;\; |A| > 0 \\
\text{no(A,B)}, & |A \cap B| = 0 \;\; \& \;\; |A| > 0 \\
\text{most(A,B)}, & |A \cap B| > |A \setminus B| \\
\text{most(A,}\neg\text{B)}, & |A \cap B| < |A \setminus B| \\
\neg\text{most(A,}\neg\text{B)}, & |A \cap B| \geq |A \setminus B| \\
\neg\text{most(A,B)}, & |A \cap B| \leq |A \setminus B| \\
\text{some(A,B)}, & |A \cap B| > 0 \\
\text{not all(A,B)} \;\; \} & |A \setminus B| > 0
\end{array}
$$

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$\Pi(\mathcal{F}_{SYLL}) =$
$\{\ \alpha_1$: $|A \setminus B| = 0\ \ \&\ \ |A| > 0,$
α_2: $|A \setminus B| > 0\ \ \&\ \ |A \cap B| > 0,$
α_3: $|A \cap B| = 0\ \ \&\ \ |A| > 0\ \}$

$\Pi(\mathcal{F}_{most}) =$
$\{\ \alpha'_1$: $|A \cap B| > |A \setminus B|,$
α'_2: $|A \cap B| = |A \setminus B|,$
α'_3: $|A \cap B| < |A \setminus B|\ \}$

- $\Pi(\mathcal{F}_{SYLLmost}) := \Pi(\mathcal{F}_{SYLL}) \wedge_{SYLL} \Pi(\mathcal{F}_{most})$
- The fragment $\mathcal{F}_{SYLLmost}$ induces the **partition** $\Pi(\mathcal{F}_{SYLLmost}) =$

$\{\ |A \cap B| > |A \setminus B| = 0,$ α''_1: All A are B and there are A's
$|A \cap B| > |A \setminus B| > 0,$ α''_2: Most but not all A's are B
$|A \cap B| = |A \setminus B| > 0,$ α''_3: Exactly half the A's are B
$0 < |A \cap B| < |A \setminus B|,$ α''_4: Most but not all A's are not B
$0 = |A \cap B| < |A \setminus B|\ \}$ α''_5: No A's are B, but there are A's

  - Compared to $\Pi(\mathcal{F}_{FOLmost})$, the sixth anchor formula α''_6 is deleted, so bit position six is deleted as well, yielding a bitstring semantics with bitstrings of length **five** instead of six.

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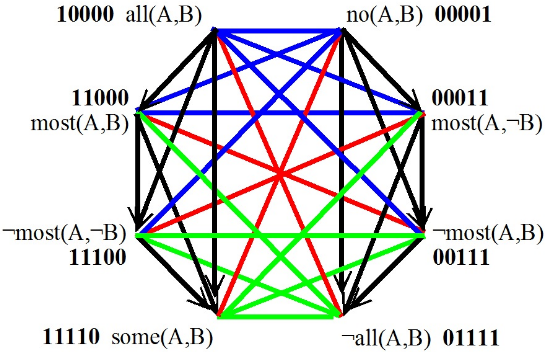
- The **bitstring semantics** $\beta_{SYLLmost}$ for the fragment $\mathcal{F}_{SYLLmost}$:

$\beta_{SYLLmost}(all(A,B))$	$=$	**10000**	$\lvert A \setminus B\rvert = 0 \;\;\&\;\; \lvert A\rvert > 0$
$\beta_{SYLLmost}(no(A,B))$	$=$	**00001**	$\lvert A \cap B\rvert = 0 \;\;\&\;\; \lvert A\rvert > 0$
$\beta_{SYLLmost}(most(A,B))$	$=$	**11000**	$\lvert A \cap B\rvert > \lvert A \setminus B\rvert$
$\beta_{SYLLmost}(most(A,\neg B))$	$=$	**00011**	$\lvert A \cap B\rvert < \lvert A \setminus B\rvert$
$\beta_{SYLLmost}(\neg most(A,\neg B))$	$=$	**11100**	$\lvert A \cap B\rvert \geq \lvert A \setminus B\rvert$
$\beta_{SYLLmost}(\neg most(A,B))$	$=$	**00111**	$\lvert A \cap B\rvert \leq \lvert A \setminus B\rvert$
$\beta_{SYLLmost}(some(A,B))$	$=$	**11110**	$\lvert A \cap B\rvert > 0$
$\beta_{SYLLmost}(not\ all(A,B))$	$=$	**01111**	$\lvert A \setminus B\rvert > 0$

- fundamental impact on the overall Aristotelian constellation:
  - from **3** relations of *contrariety* in $\mathcal{F}_{FOLLmost}$ to **6** in $\mathcal{F}_{SYLLmost}$.
  - from **3** relations of *subcontrariety* in $\mathcal{F}_{FOLLmost}$ to **6** in $\mathcal{F}_{SYLLmost}$.
  - from **6** relations of *subalternation* in $\mathcal{F}_{FOLLmost}$ to **12** in $\mathcal{F}_{SYLLmost}$.
  - from **12** relations of *unconnectedness* in $\mathcal{F}_{FOLLmost}$ to **0** in $\mathcal{F}_{SYLLmost}$



- return to a well-documented family of octagons (Lenzen 2012):
  - consists of six interlocking classical squares, but no degenerate squares.
  - illustrates well-known phenomenon of **logic-sensitivity** of Aristotelian diagrams.

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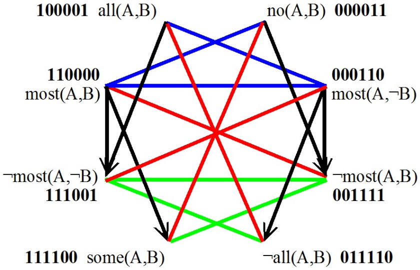
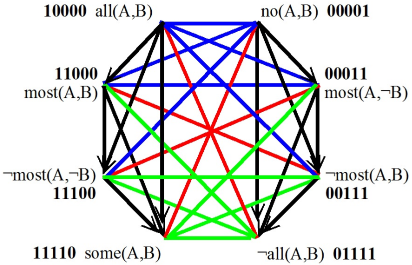
4. **Conclusion**

Conclusion

$\mathcal{F}$	$\lvert\mathcal{F}\rvert$	$\lvert\Pi(\mathcal{F})\rvert$	***Aristotelian diagram***
$\mathcal{F}_{FOL}$	4	4	degenerate square
$\mathcal{F}_{SYLL}$	4	3	classical square
$\mathcal{F}_{most}$	4	3	classical square

$\mathcal{F}$	$\lvert\mathcal{F}\rvert$	$\lvert\Pi(\mathcal{F})\rvert$	***Aristotelian diagram***
$\mathcal{F}_{FOL}$	4	4	degenerate square
$\mathcal{F}_{SYLL}$	4	3	classical square
$\mathcal{F}_{most}$	4	3	classical square
$\mathcal{F}_{FOLmost}$	8	6	**new type** of octagon
$\mathcal{F}_{SYLLmost}$	8	5	Lenzen octagon





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Thank you!

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