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Aristotelian Diagrams for Multi-Operator Formulas in Avicenna and Buridan

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- Buridan's Aristotelian octagons: relatively well-known, actual diagrams
- logical goals:
  - systematically study some natural extensions of Buridan's octagon
  - compare them in terms of their logical complexity (bitstring length)
- historical goals:
  - show that although he did not draw the actual diagram, Buridan had the logical means available to construct at least one of these extensions (historical scholarship Buridan: S. Read, G. Hughes, S. Johnston, J. Campos Benítez)
  - establish the historical priority of Al-Farabi and Avicenna with respect to Buridan's octagon and at least two of its extensions (historical scholarship Avicenna: S. Chatti, W. Hodges)
- talk based on joint research with Saloua Chatti (Université de Tunis) & Fabien Schang (HSE Moscow)

1. Some Preliminaries from Logical Geometry
2. Buridan's modal octagon = square x square
3. First extension: dodecagon = square x hexagon (Buridan/Avicenna)
4. Second extension: dodecagon = hexagon x square (Avicenna)
5. Conclusion

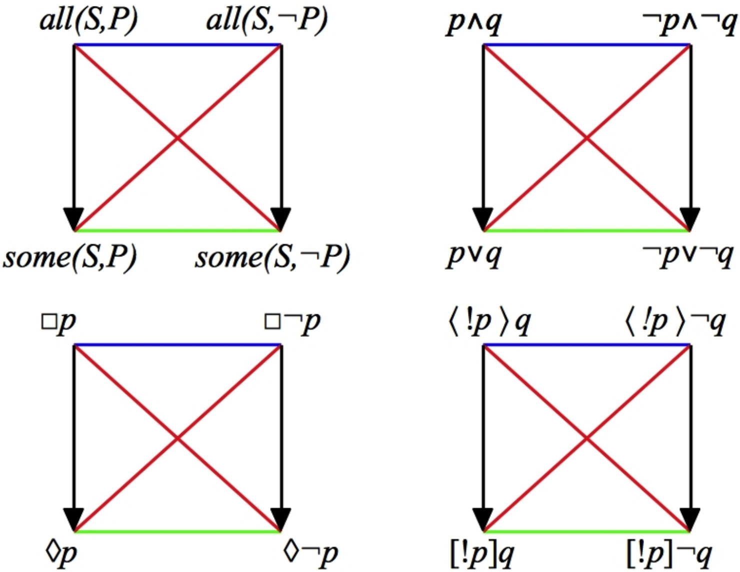
Structure of the talk

1. Some Preliminaries from Logical Geometry
2. Buridan's modal octagon = square x square
3. First extension: dodecagon = square x hexagon (Buridan/Avicenna)
4. Second extension: dodecagon = hexagon x square (Avicenna)
5. Conclusion

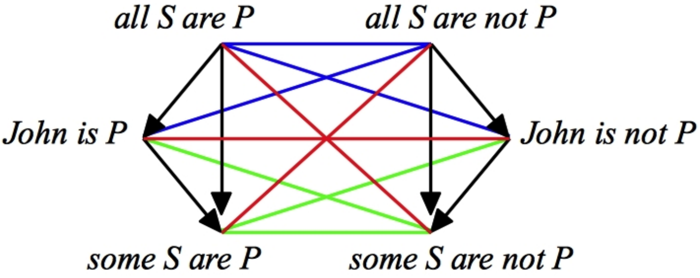
- an Aristotelian diagram visualizes some formulas and the Aristotelian relations holding between them
- definition of the Aristotelian relations: two propositions are

contradictory	iff	they cannot be true together and they cannot be false together,
contrary	iff	they cannot be true together but they can be false together,
subcontrary	iff	they can be true together but they cannot be false together,
in subalternation	iff	the first proposition entails the second but the second doesn't entail the first

Some Aristotelian squares

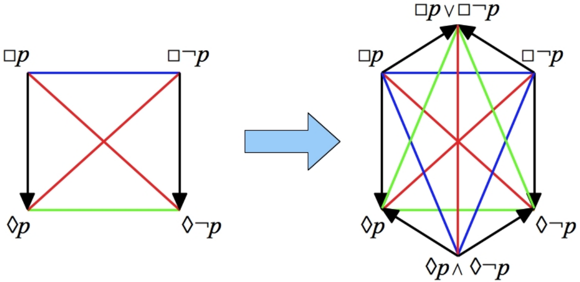


- already during the Middle Ages, philosophers used Aristotelian diagrams larger than the classical square to visualize their logical theories
- e.g. John Buridan (ca. 1295–1358): several octagons (see later)
- e.g. William of Sherwood (ca. 1200–1272), *Introductiones in Logicam* ⇒ integrating singular propositions into the classical square



Boolean closure of an Aristotelian diagram

- the smallest Aristotelian diagram that contains all contingent Boolean combinations of formulas from the original diagram
- the Boolean closure of a classical square is a Jacoby-Sesmat-Blanché hexagon (6 formulas)



- the Boolean closure of a Sherwood-Czezowski hexagon is a (3D) rhombic dodecahedron (14 formulas)

- every formula in (the Boolean closure of) an Aristotelian diagram can be represented by means of a **bitstring** = sequence of bits (0/1)
- bit-positions in a bitstring of length n correspond to '**anchor formulas**' $\alpha_1, \ldots, \alpha_n$ (obtainable from the diagram) which jointly yield a partition of logical space

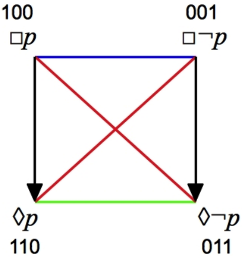
- every formula in the diagram is equivalent to a disjunction of these anchor formulas (disjunctive normal form)
- bitstrings keep track which anchor formulas occur in the disjunction and which ones do not

- bitstrings of length $n \Leftrightarrow$ size of Boolean closure is $2^n - 2$
  - disregard non-contingencies (tautology/contradiction, top/bottom)
  - bitstrings of length $3 \Leftrightarrow$ Boolean closure is $2^3 - 2 = 8 - 2 = 6$ formulas
  - bitstrings of length $4 \Leftrightarrow$ Boolean closure is $2^4 - 2 = 16 - 2 = 14$ formulas

α_1	α_2	α_3
$\Box p$	$\Diamond p \wedge \Diamond\neg p$	$\Box\neg p$
1/0	1/0	1/0

$$\Box p = \alpha_1 = 100$$
$$\Box\neg p = \alpha_3 = 001$$

$$\Diamond p \equiv \Box p \vee (\Diamond p \wedge \Diamond\neg p) = \alpha_1 \vee \alpha_2 = 110$$
$$\Diamond\neg p \equiv (\Diamond p \wedge \Diamond\neg p) \vee \Box\neg p = \alpha_2 \vee \alpha_3 = 011$$



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α_1	α_2	α_3
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1/0	1/0	1/0

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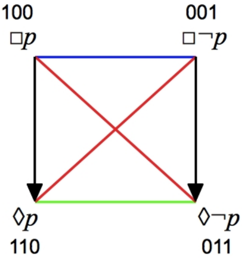
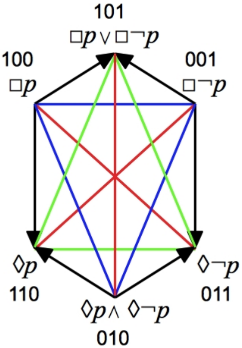
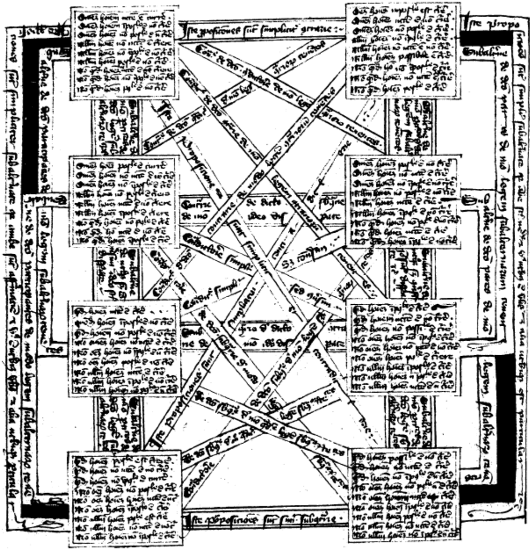


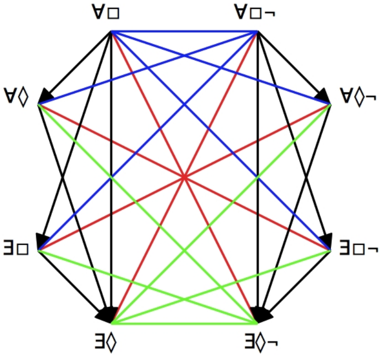


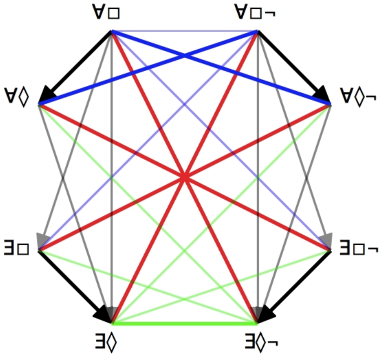
- John Buridan (ca. 1295–1358)
- *Summulae de Dialectica* (late 1330s, revisions into the 1350s)
- Vatican manuscript Pal.Lat. 994 contains several Aristotelian diagrams:
  - Aristotelian square for the usual categorical propositions (A,I,E,O) (e.g. "every human is mortal")
  - Aristotelian octagon for non-normal propositions (e.g. "every human some animal is not") (cf. regimentation of Latin)
  - Aristotelian octagon for propositions with oblique terms (e.g. "every donkey of every human is running")
  - Aristotelian octagon for modal propositions (e.g. "every human is necessarily mortal")

$$\begin{array}{rcl} \text{square} & \Rightarrow & \text{single operator} \\ \text{octagons} & \Rightarrow & \text{combined operators} \end{array}$$



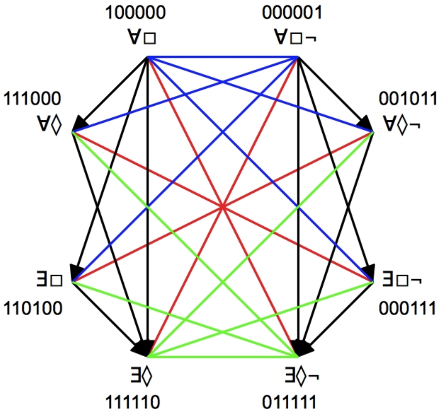
- Buridan's octagon contains the following 8 formulas:
  1. all A are necessarily B — $\forall x(\Diamond Ax \rightarrow \Box Bx)$ — $\forall\Box$
  2. all A are possibly B — $\forall x(\Diamond Ax \rightarrow \Diamond Bx)$ — $\forall\Diamond$
  3. some A are necessarily B — $\exists x(\Diamond Ax \wedge \Box Bx)$ — $\exists\Box$
  4. some A are possibly B — $\exists x(\Diamond Ax \wedge \Diamond Bx)$ — $\exists\Diamond$
  5. all A are necessarily not B — $\forall x(\Diamond Ax \rightarrow \Box\neg Bx)$ — $\forall\Box\neg$
  6. all A are possibly not B — $\forall x(\Diamond Ax \rightarrow \Diamond\neg Bx)$ — $\forall\Diamond\neg$
  7. some A are nessarily not B — $\exists x(\Diamond Ax \wedge \Box\neg Bx)$ — $\exists\Box\neg$
  8. some A are possibly not B — $\exists x(\Diamond Ax \wedge \Diamond\neg Bx)$ — $\exists\Diamond\neg$
- note: de re modality, ampliation of the subject in modal formulas
- historical precursor: **Al-Farabi** (ca. 873–950)
  - S. Chatti, 2015, *Al-Farabi on Modal Oppositions*
  - identified the 8 formulas of Buridan's octagon
  - identified only a few of the Aristotelian relations of the octagon (but all relations are deducible from the ones identified by Al-Farabi)





- unlike Buridan, Al-Farabi does not seem to have visualized his logical theorizing by means of an actual diagram
- unlike Buridan, Al-Farabi was not explicit about the issue of ampliation

α_1	α_2	α_3	α_4	α_5	α_6
$\forall\Box$	$\forall\Diamond \wedge \exists\Box \wedge \exists\Diamond\neg$	$\forall\Diamond \wedge \forall\Diamond\neg$	$\exists\Box \wedge \exists\Box\neg$	$\forall\Diamond\neg \wedge \exists\Box\neg \wedge \exists\Diamond$	$\forall\Box\neg$
1/0	1/0	1/0	1/0	1/0	1/0



- classical square (representable by bitstrings of length 3)
  $\Rightarrow$ natural extension: JSB hexagon, i.e. its Boolean closure ($6 = 2^3 - 2$)

- Buridan's modal octagon (representable by bitstrings of length 6)
  $\Rightarrow$ its Boolean closure has $2^6 - 2 = 62$ formulas $\Rightarrow$ too large!
  $\Rightarrow$ other, more 'reasonable' extensions of the octagon?

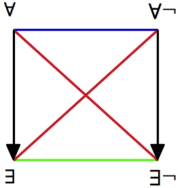
- key idea:

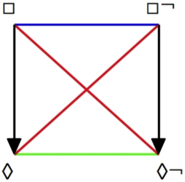
Buridan's octagon for quantified modal logic can be seen as arising out of the interaction of a quantifier square and a modality square

instead of taking the Boolean closure of the entire octagon, we can take the Boolean closure of its 'component squares'

Interaction of a quantifier square and a modality square

square $\times$ square $\Rightarrow 4 \times 4 = 16$ pairwise equivalent formulas:





$\forall\neg\Box \equiv \forall\Diamond\neg$
$\exists\neg\Box \equiv \exists\Diamond\neg$

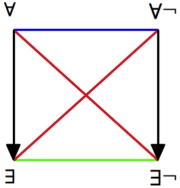
$\forall\neg\Box\neg \equiv \forall\Diamond$
$\exists\neg\Box\neg \equiv \exists\Diamond$

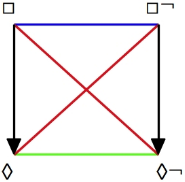
$\forall\neg\Diamond \equiv \forall\Box\neg$
$\exists\neg\Diamond \equiv \exists\Box\neg$

$\forall\neg\Diamond\neg \equiv \forall\Box$
$\exists\neg\Diamond\neg \equiv \exists\Box$

	$\Box$	$\Box\neg$	$\Diamond$	$\Diamond\neg$
$\forall$	$\forall\Box$	$\forall\Box\neg$	$\forall\Diamond$	$\forall\Diamond\neg$
$\forall\neg$	$\forall\neg\Box$	$\forall\neg\Box\neg$	$\forall\neg\Diamond$	$\forall\neg\Diamond\neg$
$\exists$	$\exists\Box$	$\exists\Box\neg$	$\exists\Diamond$	$\exists\Diamond\neg$
$\exists\neg$	$\exists\neg\Box$	$\exists\neg\Box\neg$	$\exists\neg\Diamond$	$\exists\neg\Diamond\neg$

square $\times$ square $\Rightarrow 4 \times 4 = 16$ pairwise equivalent formulas:





$\forall\neg\Box \equiv \forall\Diamond\neg$ $\exists\neg\Box \equiv \exists\Diamond\neg$

$\forall\neg\Box\neg \equiv \forall\Diamond$ $\exists\neg\Box\neg \equiv \exists\Diamond$

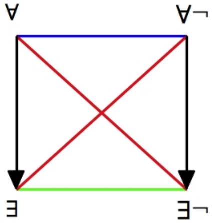
$\forall\neg\Diamond \equiv \forall\Box\neg$ $\exists\neg\Diamond \equiv \exists\Box\neg$

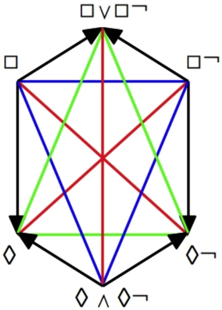
$\forall\neg\Diamond\neg \equiv \forall\Box$ $\exists\neg\Diamond\neg \equiv \exists\Box$

	$\Box$	$\Box\neg$	$\Diamond$	$\Diamond\neg$
$\forall$	$\forall\Box$	$\forall\Box\neg$	$\forall\Diamond$	$\forall\Diamond\neg$
$\forall\neg$	$\forall\neg\Box$	$\forall\neg\Box\neg$	$\forall\neg\Diamond$	$\forall\neg\Diamond\neg$
$\exists$	$\exists\Box$	$\exists\Box\neg$	$\exists\Diamond$	$\exists\Diamond\neg$
$\exists\neg$	$\exists\neg\Box$	$\exists\neg\Box\neg$	$\exists\neg\Diamond$	$\exists\neg\Diamond\neg$

A first extension of Buridan's octagon

- Buridan octagon = quantifier square × modality square
- Boolean closure of either/both components ⇒ three possibilities:
  - quantifier square × modality **hexagon**
  - quantifier **hexagon** × modality square
  - quantifier **hexagon** × modality **hexagon**





	$\Box$	$\Box\neg$	$\Diamond$	$\Diamond\neg$	$\Box \vee \Box\neg$	$\Diamond \wedge \Diamond\neg$
$\forall$	$\forall\Box$	$\forall\Box\neg$	$\forall\Diamond$	$\forall\Diamond\neg$	$\forall(\Box \vee \Box\neg)$	$\forall(\Diamond \wedge \Diamond\neg)$
$\forall\neg$	$\forall\neg\Box$	$\forall\neg\Box\neg$	$\forall\neg\Diamond$	$\forall\neg\Diamond\neg$	$\forall\neg(\Box \vee \Box\neg)$	$\forall\neg(\Diamond \wedge \Diamond\neg)$
$\exists$	$\exists\Box$	$\exists\Box\neg$	$\exists\Diamond$	$\exists\Diamond\neg$	$\exists(\Box \vee \Box\neg)$	$\exists(\Diamond \wedge \Diamond\neg)$
$\exists\neg$	$\exists\neg\Box$	$\exists\neg\Box\neg$	$\exists\neg\Diamond$	$\exists\neg\Diamond\neg$	$\exists\neg(\Box \vee \Box\neg)$	$\exists\neg(\Diamond \wedge \Diamond\neg)$

- note: $\forall(\Box \vee \Box\neg)$ should be read as: $\forall x(\Diamond Ax \rightarrow (\Box Bx \vee \Box\neg Bx))$
- 8 new formulas, but again pairwise equivalent:
  - $\forall\neg(\Box \vee \Box\neg) \equiv \forall(\Diamond \wedge \Diamond\neg)$ $\exists\neg(\Box \vee \Box\neg) \equiv \exists(\Diamond \wedge \Diamond\neg)$
  - $\forall\neg(\Diamond \wedge \Diamond\neg) \equiv \forall(\Box \vee \Box\neg)$ $\exists\neg(\Diamond \wedge \Diamond\neg) \equiv \exists(\Box \vee \Box\neg)$

A first extension of Buridan's octagon

	$\Box$	$\Box\neg$	$\Diamond$	$\Diamond\neg$	$\Box \vee \Box\neg$	$\Diamond \wedge \Diamond\neg$
$\forall$	$\forall\Box$	$\forall\Box\neg$	$\forall\Diamond$	$\forall\Diamond\neg$	$\forall(\Box \vee \Box\neg)$	$\forall(\Diamond \wedge \Diamond\neg)$
$\forall\neg$	$\forall\neg\Box$	$\forall\neg\Box\neg$	$\forall\neg\Diamond$	$\forall\neg\Diamond\neg$	$\forall\neg(\Box \vee \Box\neg)$	$\forall\neg(\Diamond \wedge \Diamond\neg)$
$\exists$	$\exists\Box$	$\exists\Box\neg$	$\exists\Diamond$	$\exists\Diamond\neg$	$\exists(\Box \vee \Box\neg)$	$\exists(\Diamond \wedge \Diamond\neg)$
$\exists\neg$	$\exists\neg\Box$	$\exists\neg\Box\neg$	$\exists\neg\Diamond$	$\exists\neg\Diamond\neg$	$\exists\neg(\Box \vee \Box\neg)$	$\exists\neg(\Diamond \wedge \Diamond\neg)$

- note: $\forall(\Box \vee \Box\neg)$ should be read as: $\forall x(\Diamond Ax \rightarrow (\Box Bx \vee \Box\neg Bx))$
- 8 new formulas, but again pairwise equivalent:
  - $\forall\neg(\Box \vee \Box\neg) \equiv \forall(\Diamond \wedge \Diamond\neg)$ $\quad$ $\exists\neg(\Box \vee \Box\neg) \equiv \exists(\Diamond \wedge \Diamond\neg)$
  - $\forall\neg(\Diamond \wedge \Diamond\neg) \equiv \forall(\Box \vee \Box\neg)$ $\quad$ $\exists\neg(\Diamond \wedge \Diamond\neg) \equiv \exists(\Box \vee \Box\neg)$
- up to logical equivalence, we arrive at $\frac{4\times 6}{2} = 12$ formulas
  $\Rightarrow$ Aristotelian **dodecagon** that extends Buridan's octagon
- more reasonable than the octagon's full Boolean closure $(8 < 12 \ll 62)$

- the ‘dodecagon’ in Buridan (ca. 1295–1358):
  - S. Read, 2015, *John Buridan on Non-Contingency Syllogisms*
  - Buridan identified the 12 formulas of the dodecagon
  - Buridan identified the Aristotelian relations of the dodecagon
- the ‘dodecagon’ in Avicenna (ca. 980–1037):
  - S. Chatti, 2015, *Les Carrés d’Avicenne*
  - Avicenna identified the 12 formulas of the dodecagon
  - Avicenna identified the Aristotelian relations of the dodecagon

Buridan:	dodecagon	=	quantifier square	$\times$	modal hexagon
Avicenna:	dodecagon	=	quantifier square	$\times$	**temporal** hexagon

formula	Buridan	Avicenna
$\exists\Box$	some A are ***necessarily*** B	some A are ***always*** B
$\forall\Diamond$	all A are ***possibly*** B	all A are ***sometimes*** B

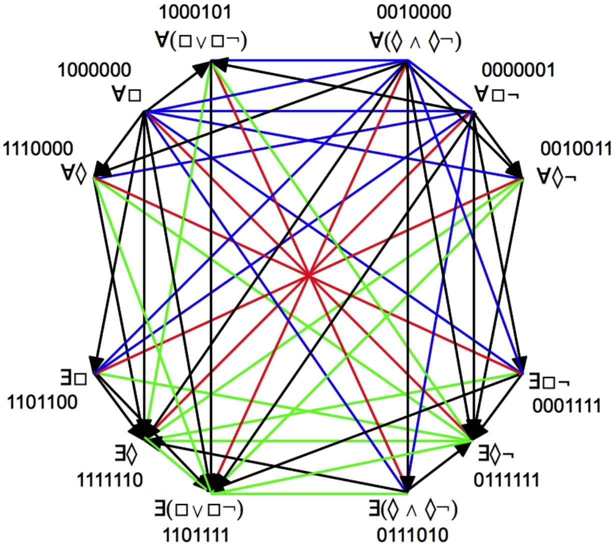
octagon (square $\times$ square) $\Rightarrow$ bitstrings of length 6

α_1	α_2	α_3	α_4	α_5	α_6
$\forall\Box$	$\forall\Diamond \wedge \exists\Box \wedge \exists\Diamond\neg$	$\forall\Diamond \wedge \forall\Diamond\neg$	$\exists\Box \wedge \exists\Box\neg$	$\forall\Diamond\neg \wedge \exists\Box\neg \wedge \exists\Diamond$	$\forall\Box\neg$
1/0	1/0	1/0	1/0	1/0	1/0

dodecagon (square $\times$ hexagon) $\Rightarrow$ bitstrings of length 7

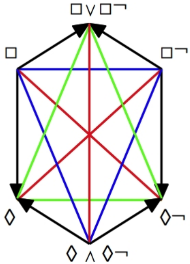
α_1	α_2	α_3	α_{4a}	α_{4b}	α_5	α_6
$\forall\Box$	$\forall\Diamond \wedge \exists\Box$ $\wedge\exists\Diamond\neg$	$\forall\Diamond \wedge \forall\Diamond\neg$	$\exists\Box \wedge \exists\Box\neg$ $\wedge\exists(\Diamond \wedge \Diamond\neg)$	$\exists\Box \wedge \exists\Box\neg$ $\wedge\forall(\Box \vee \Box\neg)$	$\forall\Diamond\neg \wedge \exists\Box\neg$ $\wedge\exists\Diamond$	$\forall\Box\neg$
1/0	1/0	1/0	1/0	1/0	1/0	1/0

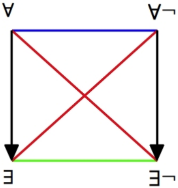
- the first extension does not fit within the octagon's Boolean closure
  - Boolean closure of the octagon: $2^6 - 2 = 62$ formulas
  - Boolean closure of the first extension: $2^7 - 2 = 126$ formulas
- quantifier does **not** distribute over modality in α_{4a} / α_{4b}


1000101
∀(□∨□¬)
0010000
∀(◊ ∧ ◊¬)
1000000
∀□
0000001
∀□¬
1110000
∀◊
0010011
∀◊¬
∃□
1101100
∃□¬
0001111
∃◊
1111110
∃◊¬
0111111
∃(□∨□¬)
1101111
∃(◊ ∧ ◊¬)
0111010

A second extension of Buridan's octagon

- Buridan: octagon = quantifier square $\times$ modality square
- first extension: take Boolean closure of the second square
  $\Rightarrow$ dodecagon = quantifier square $\times$ modality **hexagon**
- second extension: take Boolean closure of the first square
  $\Rightarrow$ dodecagon = modality **hexagon** $\times$ quantifier square
  - also switch the roles of quantifiers and modalities
  - from de re modalities to de dicto modalities





	$\forall$	$\forall\neg$	$\exists$	$\exists\neg$
$\Box$	$\Box\forall$	$\Box\forall\neg$	$\Box\exists$	$\Box\exists\neg$
$\Box\neg$	$\Box\neg\forall$	$\Box\neg\forall\neg$	$\Box\neg\exists$	$\Box\neg\exists\neg$
$\Diamond$	$\Diamond\forall$	$\Diamond\forall\neg$	$\Diamond\exists$	$\Diamond\exists\neg$
$\Diamond\neg$	$\Diamond\neg\forall$	$\Diamond\neg\forall\neg$	$\Diamond\neg\exists$	$\Diamond\neg\exists\neg$
$\Box \vee \Box\neg$	$(\Box \vee \Box\neg)\forall$	$(\Box \vee \Box\neg)\forall\neg$	$(\Box \vee \Box\neg)\exists$	$(\Box \vee \Box\neg)\exists\neg$
$\Diamond \wedge \Diamond\neg$	$(\Diamond \wedge \Diamond\neg)\forall$	$(\Diamond \wedge \Diamond\neg)\forall\neg$	$(\Diamond \wedge \Diamond\neg)\exists$	$(\Diamond \wedge \Diamond\neg)\exists\neg$

- note: $(\Box \vee \Box\neg)\forall$ should be read as: $\Box\forall \vee \Box\neg\forall$ ($\equiv \Box\forall \vee \Box\exists\neg$)
- pairwise equivalent: $\frac{6\times4}{2} = 12$ formulas $\Rightarrow$ second dodecagon extension

	$\forall$	$\forall\neg$	$\exists$	$\exists\neg$
$\Box$	$\Box\forall$	$\Box\forall\neg$	$\Box\exists$	$\Box\exists\neg$
$\Box\neg$	$\Box\neg\forall$	$\Box\neg\forall\neg$	$\Box\neg\exists$	$\Box\neg\exists\neg$
$\Diamond$	$\Diamond\forall$	$\Diamond\forall\neg$	$\Diamond\exists$	$\Diamond\exists\neg$
$\Diamond\neg$	$\Diamond\neg\forall$	$\Diamond\neg\forall\neg$	$\Diamond\neg\exists$	$\Diamond\neg\exists\neg$
$\Box \vee \Box\neg$	$(\Box \vee \Box\neg)\forall$	$(\Box \vee \Box\neg)\forall\neg$	$(\Box \vee \Box\neg)\exists$	$(\Box \vee \Box\neg)\exists\neg$
$\Diamond \wedge \Diamond\neg$	$(\Diamond \wedge \Diamond\neg)\forall$	$(\Diamond \wedge \Diamond\neg)\forall\neg$	$(\Diamond \wedge \Diamond\neg)\exists$	$(\Diamond \wedge \Diamond\neg)\exists\neg$

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- the 'dodecagon' in Avicenna (ca. 980–1037):
  - S. Chatti, 2014, *Avicenna on Possibility and Necessity*
  - Avicenna identified the 12 formulas of this second dodecagon
  - Avicenna identified the Aristotelian relations holding between them

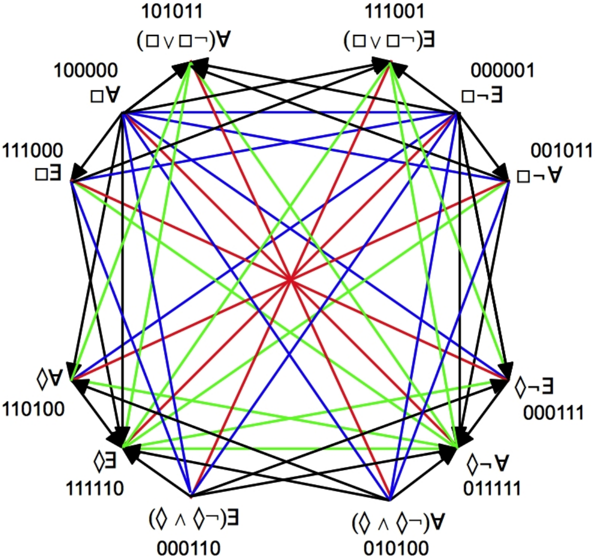
octagon (square × square) ⇒ bitstrings of length 6

α_1	α_2	α_3	α_4	α_5	α_6
$\forall\Box$	$\forall\Diamond \wedge \exists\Box \wedge \exists\Diamond\neg$	$\forall\Diamond \wedge \forall\Diamond\neg$	$\exists\Box \wedge \exists\Box\neg$	$\forall\Diamond\neg \wedge \exists\Box\neg \wedge \exists\Diamond$	$\forall\Box\neg$
1/0	1/0	1/0	1/0	1/0	1/0

dodecagon (hexagon × square) ⇒ bitstrings of length 6

α_1	α_2	α_3	α_4	α_5	α_6
$\Box\forall$	$\Box\exists \wedge \Diamond\forall \wedge \Diamond\exists\neg$	$\Box\exists \wedge \Box\exists\neg$	$\Diamond\forall \wedge \Diamond\forall\neg$	$\Box\exists\neg \wedge \Diamond\forall\neg \wedge \Diamond\exists$	$\Box\forall\neg$
1/0	1/0	1/0	1/0	1/0	1/0

- anchor formulas are the same
  (except that quantifiers and modalities are switched)
- second extension of Buridan's octagon
  remains within that octagon's Boolean closure



101011
(□ ∨ □¬)∀
111001
(□ ∨ □¬)∃
100000
□∀
000001
□¬∃
111000
□∃
001011
□¬∀
◊∀
110100
◊¬∃
000111
◊∃
111110
◊¬∀
011111
(◊ ∧ ◊¬)∃
000110
(◊ ∧ ◊¬)∀
010100

Summary

- natural extension from a technical (and historical?) perspective:
  - take Boolean closure of both square components
  - so we get hexagon $\times$ hexagon $\Rightarrow \frac{6\times 6}{2} = 18$ formulas
  - e.g. "some but not all men are contingently philosophers"

- overview:

Buridan	8-gon	quantifier square	$\times$	modality square	6
"Al-Farabi"	8-gon	quantifier square	$\times$	modality square	6
"Buridan"	12-gon	quantifier square	$\times$	modality hexagon	7
"Avicenna"	12-gon	quantifier square	$\times$	temporal hexagon	7
"Avicenna"	12-gon	modality hexagon	$\times$	quantifier square	6
???	18-gon	quantifier hexagon	$\times$	modal hexagon	7

Thank you!

More info: www.logicalgeometry.org