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An Introduction to Logical Geometry

Hans Smessaert and Lorenz Demey

Structure of the talk

1. General introduction
   - Central aim of Logical Geometry
   - Bitstrings in Logical Geometry
   - Aristotelian relations and diagrams
   - Logical Geometry and Formal Semantics
2. Information in Aristotelian Diagrams
   - Problems with the Aristotelian geometry
   - Two new geometries
   - Information levels of logical relations and Unconnectedness
3. The Logical Geometry of the Rhombic Dodecahedron
   - Aristotelian subdiagrams
   - The Rhombic Dodecahedron of Oppositions RDH
   - (Families of) Sigma-structures: the CO-perspective
   - Complementarities between families of Sigma-structures
4. Conclusions

Central aim of Logical Geometry

The central aim of Logical Geometry (**www.logicalgeometry.org**) is

- to develop an *interdisciplinary framework*
- for the study of *geometrical representations*
- in the analysis of *logical relations*.

More in particular:

- we analyse the **logical relations** of opposition, implication and duality between expressions in various logical, linguistic and conceptual systems.
- we study abstract **geometrical representations** of these relations as well as their visualisation by means of 2D and 3D diagrams.
- we develop an **interdisciplinary framework** integrating insights from logic, formal semantics, algebra, group theory, lattice theory, computer graphics, cognitive psychology, information visualisation and diagrams design.

- Smessaert (1993). *The Logical Geometry of Comparison and Quantification. A cross-categorial analysis of Dutch determiners and aspectual adverbs*.
- World Congress on the Square of Opposition (Jean-Yves Béziau)
  - Square 2007: Montreux, Switzerland
  - Square 2010: Corte, Corsica
  - Square 2012: Beirut, Lebanon
  - Square 2014: Vatican, Roma
- Alessio Moretti (2009). *The geometry of logical opposition*. PhD in logic, University of Neuchâtel, Switzerland
- International Conference on the Theory and Application of Diagrams
  - Diagrams 2012: Canterbury, UK
  - Diagrams 2014: Melbourne, Australia

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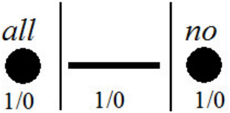
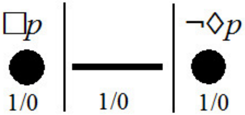
4. Conclusions

- Bitstrings are sequences of bits (0/1) that encode the denotations of formulas or expressions from:
  - **logical systems**: e.g. classical propositional logic, first-order logic, modal logic and public announcement logic
  - **lexical fields**: e.g. comparative quantification, subjective quantification, color terms and set inclusion relations
- Each bit provides an answer to a (binary) meaningful question (analysis of generalized quantifiers as sets of sets).
- Each question concerns a component (point or interval) of a scalar structure creating a partition of logical space:





- In Predicate Logic/GQT: Is R(A,B) true if

  $A \subseteq B$ yes/no

  $A \not\subseteq B$ and $A \cap B \neq \emptyset$ yes/no

  $A \cap B = \emptyset$ yes/no

- In Modal Logic: Is φ true if

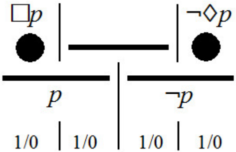
  p is true in all possible worlds? yes/no

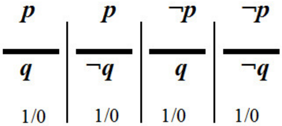
  p is true in some but not in all possible worlds? yes/no

  p is true in no possible worlds? yes/no

Modal Logic	GQT	level 1/0	level 2/3	GQT	Modal Logic
necessary ($\Box p$)	*all*	100	011	*not all*	*not necessary* ($\neg\Box p$)
contingent ($\neg\Box p \wedge \Diamond p$)	*some but not all*	010	101	*no or all*	*not contingent* ($\Box p \vee \neg\Diamond p$)
impossible ($\neg\Diamond p$)	*no*	001	110	*some*	*possible* ($\Diamond p$)
contradiction ($\Box p \wedge \neg\Box p$)	*some and no*	000	111	*some or no*	*tautology* ($\Box p \vee \neg\Box p$)

- In Modal Logic S5: Is φ true if:
  - p is true in all possible worlds? yes/no
  - p is true in the actual world but not in all possible worlds? yes/no
  - p is true in some possible worlds but not in the actual world? yes/no
  - p is true in no possible worlds? yes/no
- In Propositional Logic: Is φ true if:
  - p is true and q is true? yes/no
  - p is true and q is false? yes/no
  - p is false and q is true? yes/no
  - p is false and q is false? yes/no





$2^3 = 8$ bitstrings of length 3 $\rightsquigarrow$ $2^4 = 16$ bitstrings of length 4

Modal Logic S5	Propositional Logic	bitstrings level 1	bitstrings level 3	Propositional Logic	Modal Logic S5
$\Box p$	$p \wedge q$	1000	0111	$\neg(p \wedge q)$	$\neg\Box p$
$\neg\Box p \wedge p$	$\neg(p \rightarrow q)$	0100	1011	$p \rightarrow q$	$\Box p \vee \neg p$
$\Diamond p \wedge \neg p$	$\neg(p \leftarrow q)$	0010	1101	$p \leftarrow q$	$\neg\Diamond p \vee p$
$\neg\Diamond p$	$\neg(p \vee q)$	0001	1110	$p \vee q$	$\Diamond p$

Modal Logic S5	Propositional Logic	bitstrings level 2/0	bitstrings level 2/4	Propositional Logic	Modal Logic S5
p	p	1100	0011	$\neg p$	$\neg p$
$\Box p \vee (\Diamond p \wedge \neg p)$	q	1010	0101	$\neg q$	$\neg\Diamond p \vee (\neg\Box p \wedge p)$
$\Box p \vee \neg\Diamond p$	$p \leftrightarrow q$	1001	0110	$\neg(p \leftrightarrow q)$	$\neg\Box p \wedge \Diamond p$
$\Box p \wedge \neg\Box p$	$p \wedge \neg p$	0000	1111	$p \vee \neg p$	$\Box p \vee \neg\Box p$

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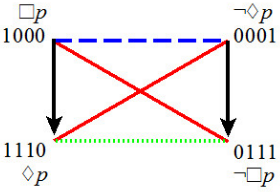
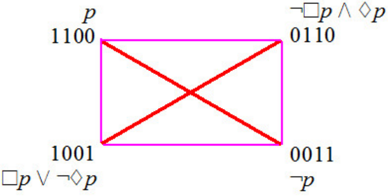
- Informally, two formulas are:
  - contradictory iff they cannot be true together and cannot be false together
  - contrary iff they cannot be true together, but can be false together
  - subcontrary iff they can be true together, but cannot be false together
  - in subalternation iff the first logically entails the second, but not vice versa
- Running example: the modal logic S5:



classical square
2 × L1-L3



degenerate square
2 × L2-L2

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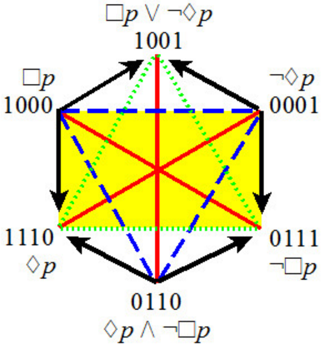
- Formally (relative to a logical system S), two **formulas** φ, ψ are

contradictory	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$
in subalternation	iff	$\mathsf{S} \models \varphi \to \psi$	and	$\mathsf{S} \not\models \psi \to \varphi$

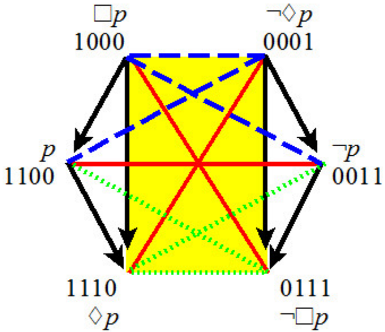
- In terms of bitstrings, two **bitstrings** b_1 and b_2 are

contradictory	iff	$b_1 \wedge b_2 = 0000$	and	$b_1 \vee b_2 = 1111$
contrary	iff	$b_1 \wedge b_2 = 0000$	and	$b_1 \vee b_2 \neq 1111$
subcontrary	iff	$b_1 \wedge b_2 \neq 0000$	and	$b_1 \vee b_2 = 1111$
in subalternation	iff	$b_1 \wedge b_2 = b_1$	and	$b_1 \vee b_2 \neq b_1$

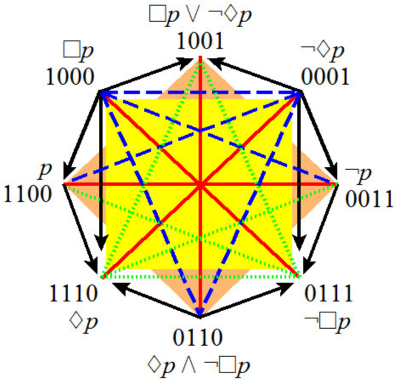
- φ and ψ stand in some Aristotelian relation (defined for S) iff $\beta(\varphi)$ and $\beta(\psi)$ stand in that same relation (defined for bitstrings).
- β maps formulas from S to bitstrings, preserving Aristotelian structure (Representation Theorem for finite Boolean algebras)



Jacoby-Sesmat-Blanché hexagon
2 × L1-L3
1 × L2-L2



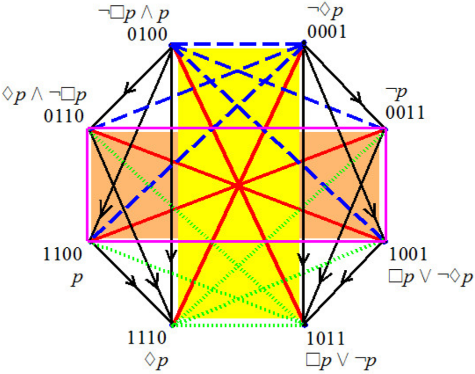
Sherwood-Czezowski hexagon
2 × L1-L3
1 × L2-L2



Béziau octagon

2 × L1-L3

2 × L2-L2



Buridan octagon

2 × L1-L3

2 × L2-L2

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$$
\begin{array}{ccccc}
 & & \text{formation rules} & & \\
 & & \Downarrow & & \\
\text{expression} & \Rightarrow & \textbf{syntactic representation} & \Rightarrow & \text{world/mind} \\
 & & \Downarrow & & \\
 & & \text{syntactic properties} & &
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \text{formation rules} & & \\
 & & \Downarrow & & \\
\text{expression} & \Rightarrow & \textbf{semantic representation} & \Rightarrow & \text{world/mind} \\
 & & \Downarrow & & \\
 & & \text{semantic properties} & &
\end{array}
$$

formation rules

$\Downarrow$

expression $\Rightarrow$ **Montague Intensional Type Logic**
Generalized Quantifier Theory $\Rightarrow$ world/mind

$\Downarrow$

semantic properties

formation rules

$\Downarrow$

sets of expressions $\Rightarrow$ **Discourse Representation Structures (DRSs)** $\Rightarrow$ world/mind

$\Downarrow$

semantic properties/relations

formation rules

$\Downarrow$

expression $\Rightarrow$ **Bitstring Representation** $\Rightarrow$ world/mind

$\Downarrow$

semantic properties

formation rules

$\Downarrow$

sets of expressions $\Rightarrow$ **Aristotelian Diagrams** $\Rightarrow$ world/mind

$\Downarrow$

semantic properties/relations

- recall the Aristotelian geometry: φ and ψ are said to be

contradictory	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
in subalternation	iff	$S \models \varphi \to \psi$	and	$S \not\models \psi \to \varphi$

- problems with the Aristotelian geometry:
  - not mutually exclusive: e.g. $\bot$ and p are contrary *and* subaltern (problem disappears if we restrict to contingent formulas)
  - not exhaustive: e.g. p and $\Diamond p \wedge \Diamond\neg p$ are in no Arist. relation at all (if φ is contingent, then φ is in no Arist. relation to itself)
  - conceptual confusion: true/false together vs truth propagation
    - 'together' $\rightsquigarrow$ symmetrical relations (undirected)
    - 'propagation' $\rightsquigarrow$ asymmetrical relations (directed)

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Structure of the talk

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- the **opposition geometry (OG)**: φ and ψ are

contradictory	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
non-contradictory	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \not\models \neg(\neg\varphi \wedge \neg\psi)$

- the **implication geometry (IG)**: φ and ψ are in

bi-implication	iff	$S \models \varphi \rightarrow \psi$	and	$S \models \psi \rightarrow \varphi$
left-implication	iff	$S \models \varphi \rightarrow \psi$	and	$S \not\models \psi \rightarrow \varphi$
right-implication	iff	$S \not\models \varphi \rightarrow \psi$	and	$S \models \psi \rightarrow \varphi$
non-implication	iff	$S \not\models \varphi \rightarrow \psi$	and	$S \not\models \psi \rightarrow \varphi$

- opposition relations: being true/false together $\varphi \wedge \psi$ and $\neg\varphi \wedge \neg\psi$
- implication relations: truth propagation $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$

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- OG and IG jointly solve the problems of the Aristotelian geometry:
  - each pair of formulas stands in exactly one opposition relation
  - each pair of formulas stands in exactly one implication relation
  - no more conceptual confusion
- conceptual independence, yet clear relationship (symmetry breaking):

$$\begin{array}{lcl} \mathsf{CD}(\varphi,\psi) & \Leftrightarrow & \mathsf{BI}(\psi,\neg\varphi) \\ \mathsf{C}(\varphi,\psi) & \Leftrightarrow & \mathsf{LI}(\psi,\neg\varphi) \\ \mathsf{SC}(\varphi,\psi) & \Leftrightarrow & \mathsf{RI}(\psi,\neg\varphi) \\ \mathsf{NCD}(\varphi,\psi) & \Leftrightarrow & \mathsf{NI}(\psi,\neg\varphi) \end{array}$$

- Correia: two philosophical traditions in Aristotle scholarship
  - square as a theory of negation — commentaries on *De Interpretatione*
  - square as a theory of consequence — commentaries on *Prior Analytics*

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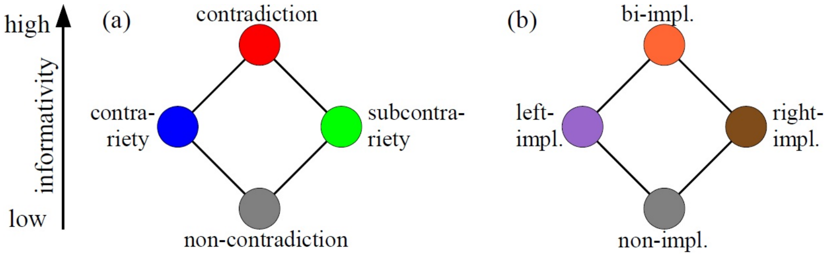
Information levels of logical relations

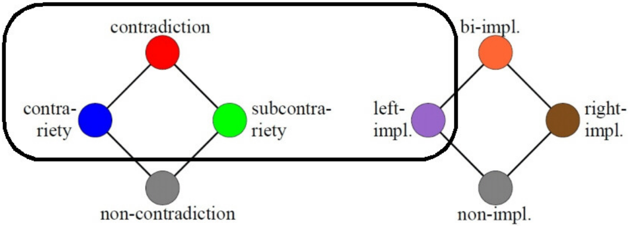
- informativity of a relation holding between φ and ψ is inversely correlated with the number of states (models) it is compatible with
- informativity of the opposition and implication relations:



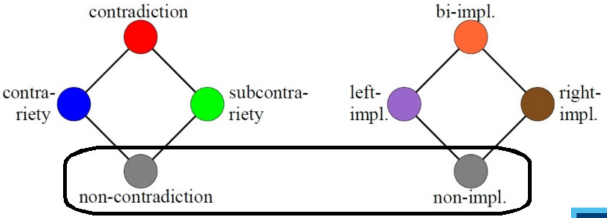
Informativity of the Aristotelian Geometry

- Aristotelian geometry: **hybrid** between
  - opposition geometry: contradiction, contrariety, subcontrariety
  - implication geometry: left-implication (subalternation)

- these relations are highly informative (in their geometries)



- given any two formulas:
  - they stand in exactly one opposition relation R
  - they stand in exactly one implication relation S
- informative relation in OG combines with uninformative relation in IG and vice versa
- exception = NCD + NI = **unconnectedness** (logical independence)
  - no Aristotelian relation at all (non-exhaustiveness of AG)
  - combination of the two *least informative* relations
  - Aristotelian gap = information gap



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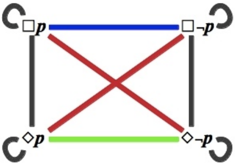
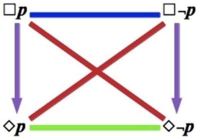
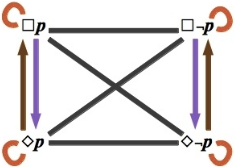
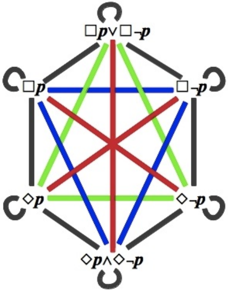
- no unconnectedness in the classical Aristotelian square

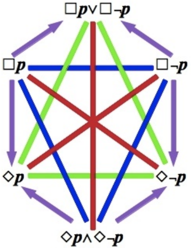


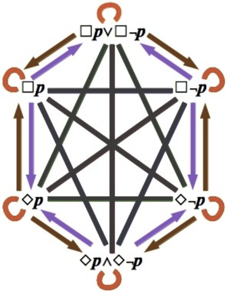




- no unconnectedness in the Jacoby-Sesmat-Blanché hexagon







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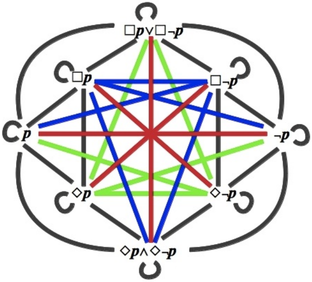
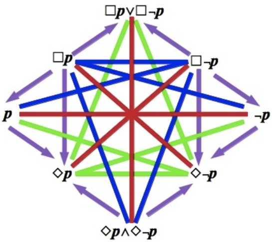
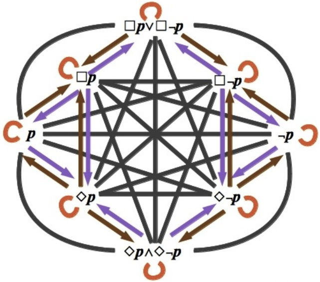
- unconnectedness in the Béziau octagon
- e.g. p and $\Diamond p \wedge \Diamond\neg p$ are unconnected







- logical geometry: Aristotelian square of oppositions and its extensions

- the Aristotelian square is highly informative:
  - Aristotelian geometry is hybrid: *maximize* informativity
    $\Rightarrow$ applies to *all* Aristotelian diagrams
  - avoid unconnectedness: *minimize* uninformativity
    $\Rightarrow$ some Aristotelian diagrams succeed better than others
    - classical square, JSB hexagon, SC hexagon don't have unconnectedness
    - Béziau octagon (and many other diagrams) do have unconnectedness

- Q: what about the JSB hexagon, SC hexagon, etc.?
  - equally informative as the square
  - yet less widely known...
- A: requires yet another geometry: duality

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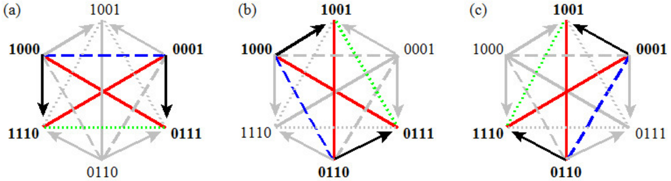
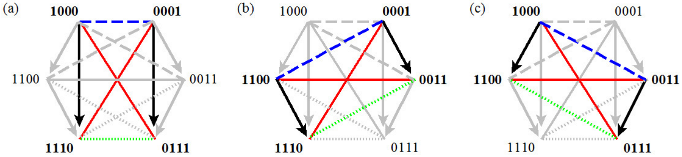
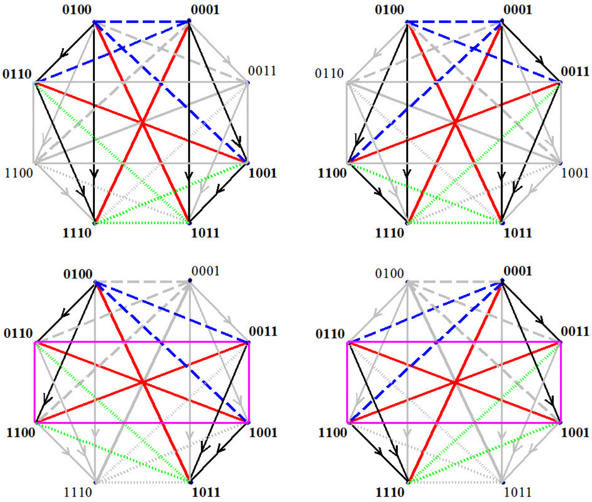
3 squares embedded in (strong) Jacoby-Sesmat-Blanché hexagon (JSB)



3 squares embedded in Sherwood-Czezowski hexagon (SC)



4 hexagons embedded in Buridan octagon



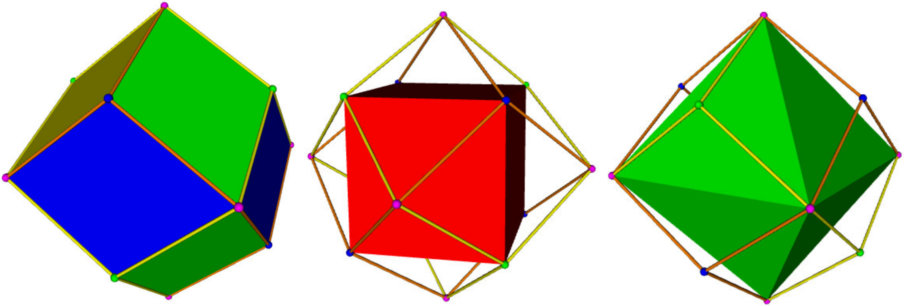
Internal structure of bigger/3D Aristotelian diagrams ? Some initial results:

- 4 **weak** JSB-hexagons in logical cube (Moretti-Pellissier)
- 6 **strong** JSB hexagons in bigger 3D structure with 14 formulas/vertices
  - tetra-hexahedron (Sauriol)
  - tetra-icosahedron (Moretti-Pellissier)
  - nested tetrahedron (Lewis, Dubois-Prade)
  - **rhombic dodecahedron = RDH** (Smessaert-Demey) => joint work

Greater complexity of RDH $\rightsquigarrow$ exhaustive analysis of internal structure ??
Main aim of this talk $\rightsquigarrow$ tools and techniques for such an analysis

- examine larger substructures (octagon, decagon, dodecagon, ...)
- distinguish families of substructures (strong JSB, weak JSB, ...)
- establish the exhaustiveness of the typology

cube	+	octahedron	=	cuboctahedron	$\overset{dual}{\Longrightarrow}$	**rhombic dodecahedron**
Platonic		Platonic		Archimedean		Catalan
6 faces		8 faces		14 faces		**12 faces**
8 vertices		6 vertices		12 vertices		**14 vertices**

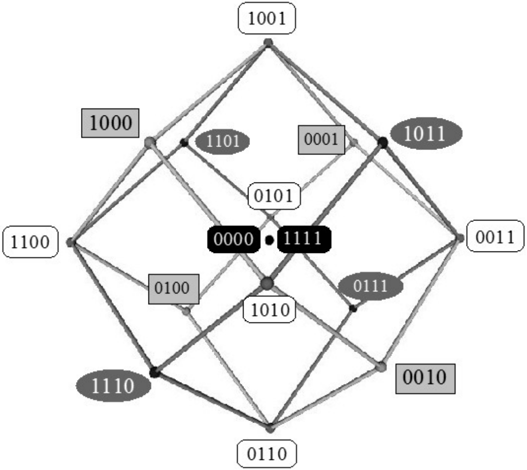


14 vertices of RDH decorated with 14 bitstrings of length 4

Modal Logic S5	Propositional Logic	bitstrings level 1	bitstrings level 3	Propositional Logic	Modal Logic S5
$\Box p$	$p \wedge q$	1000	0111	$\neg(p \wedge q)$	$\neg\Box p$
$\neg\Box p \wedge p$	$\neg(p \rightarrow q)$	0100	1011	$p \rightarrow q$	$\Box p \vee \neg p$
$\Diamond p \wedge \neg p$	$\neg(p \leftarrow q)$	0010	1101	$p \leftarrow q$	$\neg\Diamond p \vee p$
$\neg\Diamond p$	$\neg(p \vee q)$	0001	1110	$p \vee q$	$\Diamond p$

Modal Logic S5	Propositional Logic	bitstrings level 2/0	bitstrings level 2/4	Propositional Logic	Modal Logic S5
p	p	1100	0011	$\neg p$	$\neg p$
$\Box p \vee (\Diamond p \wedge \neg p)$	q	1010	0101	$\neg q$	$\neg\Diamond p \vee (\neg\Box p \wedge p)$
$\Box p \vee \neg\Diamond p$	$p \leftrightarrow q$	1001	0110	$\neg(p \leftrightarrow q)$	$\neg\Box p \wedge \Diamond p$
$\Box p \wedge \neg\Box p$	$p \wedge \neg p$	0000	1111	$p \vee \neg p$	$\Box p \vee \neg\Box p$

cube = 4 × L1 + 4 × L3 / **octahedron** = 6 × L2 / **center** = L0 + L4



Bitstrings have been used to encode

- **logical systems**: e.g. classical propositional logic, first-order logic, modal logic and public announcement logic
- **lexical fields**: e.g. comparative quantification, subjective quantification, color terms and set inclusion relations

Contradiction relation is visualized using the **central symmetry** of RDH:

- contradictory bitstrings (e.g. 1100 and 0011) occupy diametrically opposed vertices
- the negation of a bitstring is located at a maximal (Euclidean) distance from that bitstring.
- nearly all Aristotelian diagrams discussed in the literature observe central symmetry ("contradictories are diagonals")

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Bitstrings/formulas come in **pairs of contradictories (PCD)**
Key notion in describing RDH is $\sigma_{\mathbf{n}}$-**structure**.

- A σ_n-structure consists of n PCDs
- A σ_n-structure is visualized by means of a centrally symmetrical diagram
- Examples

a square has 2 PCDs	$\Rightarrow$	σ_2-structure
a hexagon has 3 PCDs	$\Rightarrow$	σ_3-structure
an octagon has 4 PCDs	$\Rightarrow$	σ_4-structure
a cube has 4 PCDs	$\Rightarrow$	σ_4-structure

Remarks

- 1 σ-structure may correspond to different σ-diagrams:
  - alternative 2D visualisations
  - 2D versus 3D representations
- All σ-structures have an even number of formulas/bitstrings
- Nearly all Aristotelian diagrams in the literature are σ-structures

Original question of Aristotelian subdiagrams ("How many smaller diagrams inside bigger diagram?") can now be reformulated in terms of σ-structures.

- For n $\leq$ k, the nummer of σ_n-structures embedded in a σ_k-structure can be calculated as the number of combinations of n PCDs out of k by means of the simple combinatorial formula: $\binom{k}{n} = \frac{k!}{n!(k-n)!}$
- This combinatorial technique $\rightsquigarrow$ recover well-known results:
  - #squares (σ_2) inside a hexagon (σ_3) is $\binom{3}{2}$: $\frac{3!}{2!(1)!} = \frac{6}{2} = 3$
  - #hexagons (σ_3) inside octagon (σ_4) is $\binom{4}{3}$: $\frac{4!}{3!(1)!} = \frac{24}{6} = 4$
- This combinatorial technique $\rightsquigarrow$ obtain new results for RDH:
  - RDH contains 14 vertices, hence 7 PCDs $\rightsquigarrow$ RDH = σ_7-structure
  - Calculate the number of σ_n-structures inside a σ_7-structure as the number of combinations of n PCDs out of 7: $\binom{7}{n} = \frac{7!}{n!(7-n)!}$

σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
$\binom{7}{0}$	$\binom{7}{1}$	$\binom{7}{2}$	$\binom{7}{3}$	$\binom{7}{4}$	$\binom{7}{5}$	$\binom{7}{6}$	$\binom{7}{7}$
$\frac{7!}{0!(7)!}$	$\frac{7!}{1!(6)!}$	$\frac{7!}{2!(5)!}$	$\frac{7!}{3!(4)!}$	$\frac{7!}{4!(3)!}$	$\frac{7!}{5!(2)!}$	$\frac{7!}{6!(1)!}$	$\frac{7!}{7!(0)!}$
$\frac{5040}{1\times5040}$	$\frac{5040}{1\times720}$	$\frac{5040}{2\times120}$	$\frac{5040}{6\times24}$	$\frac{5040}{24\times6}$	$\frac{5040}{120\times2}$	$\frac{5040}{720\times1}$	$\frac{5040}{5040\times1}$
1	7	21	35	35	21	7	1

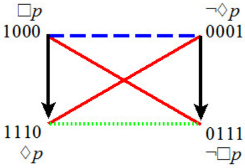
- 3 squares in 1 JSB $\times$ 6 JSB in RDH = 18 squares in RDH.
  Remaining 3 ?? Unconnected/degenerate squares
- 6 strong JSB + 4 weak JSB = 10 hexagons in RDH.
  Remaining 25 ?? Sherwood-Czezowski. Others ? Unconnected4/12.
- symmetry/mirror image ? Complementarity:
  $\#\sigma_0 = \#\sigma_7$, $\#\sigma_1 = \#\sigma_6$, $\#\sigma_2 = \#\sigma_5$, $\#\sigma_3 = \#\sigma_4$,

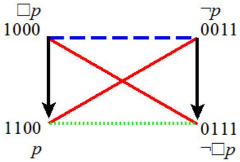
rhombic dodecahedron (RDH)	=	cube (C)	+	octahedron (O)
σ_7	=	σ_4	+	σ_3
7 PCDs	=	4 PCDs L1-L3	+	3 PCDs L2-L2

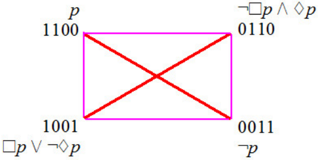
Construct a principled typology of families of σ-structures inside RDH.

- $\sigma_n = n$ out of the 7 PCDs of RDH
- $\sigma_n = [k$ out of the 4 PCDs of $C] + [\ell$ out of the 3 PCDs of $O]$
- **CO-perspective**: every class of σ_n-structures can be subdivided into families of the form $C_k O_l$, for $0 \leq k \leq 4$; $0 \leq \ell \leq 3$ and $k + \ell = n$.
- For example, the cube C is $C_4 O_0$, and the octahedron O is $C_0 O_3$.
- The number of $C_k O_\ell$-structures inside RDH ($C_4 O_3$) can be calculated as $\binom{4}{k}\binom{3}{\ell}$.

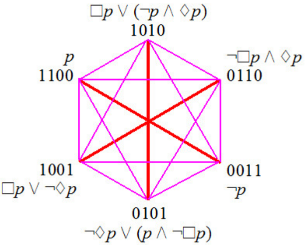
σ_2	$=$	C_2O_0	$+$	C_1O_1	$+$	C_0O_2
$\binom{7}{2}$		$\binom{4}{2}\binom{3}{0}$		$\binom{4}{1}\binom{3}{1}$		$\binom{4}{0}\binom{3}{2}$
21	$=$	6	$+$	12	$+$	3
squares		classical balanced 2×L1/2×L3		classical unbalanced 1×L1/2×L2/1×L3		degenerated (balanced) 4×L2

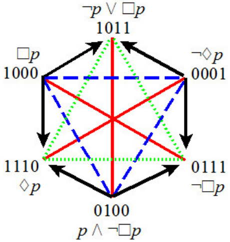


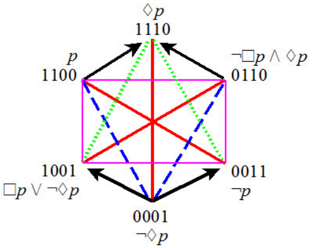




σ_3	$=$	C_0O_3	$+$	C_3O_0	$+$	C_1O_2	$+$	C_2O_1
$\binom{7}{3}$		$\binom{4}{0}\binom{3}{3}$		$\binom{4}{3}\binom{3}{0}$		$\binom{4}{1}\binom{3}{2}$		$\binom{4}{2}\binom{3}{1}$
35	$=$	1	$+$	4	$+$	12	$+$	18
hexagons		degener. U12		weak JSB		degener. U4		strong JSB Sher-Czez







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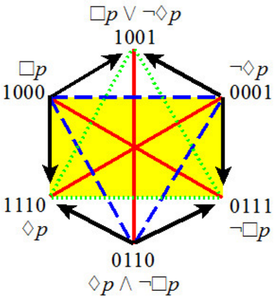
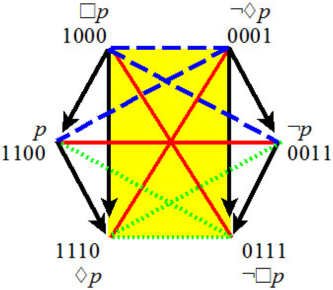
$18 \times C_2O_1 = 6 \times C_2O_1a + 12 \times C_2O_1b$

strong JSB — Sherwood-Czezowski





- CO-perspective: no distinction strong JSB *vs* Sherwood-Czezowski
- isomorphism perspective: no distinction strong JSB *vs* weak JSB

Families of σ_n-structures: overview

σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
			C_0O_3	C_4O_0			
			1	1			
	C_1O_0	C_0O_2	C_3O_0	C_1O_3	C_4O_1	C_3O_3	
	4	3	4	4	3	4	
C_0O_0	C_0O_1	C_2O_0	C_2O_1a	C_2O_2a	C_2O_3	C_4O_2	C_4O_3
1	3	6	6	6	6	3	1
		C_1O_1	C_2O_1b	C_2O_2b	C_3O_2		
		12	12	12	12		
			C_1O_2	C_3O_1			
			12	12			
1	7	21	35	35	21	7	1

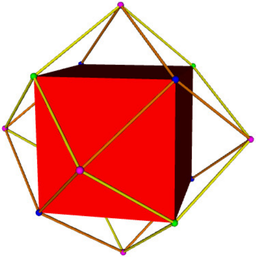
1. General introduction
   - Central aim of Logical Geometry
   - Bitstrings in Logical Geometry
   - Aristotelian relations and diagrams
   - Logical Geometry and Formal Semantics
2. Information in Aristotelian Diagrams
   - Problems with the Aristotelian geometry
   - Two new geometries
   - Information levels of logical relations and Unconnectedness
3. The Logical Geometry of the Rhombic Dodecahedron
   - Aristotelian subdiagrams
   - The Rhombic Dodecahedron of Oppositions RDH
   - (Families of) Sigma-structures: the CO-perspective
   - Complementarities between families of Sigma-structures
4. Conclusions

Complementarities between families of σ_n-structures

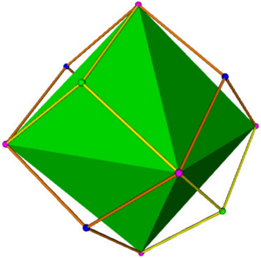
Fundamental complementarity between σ-structures inside RDH

- $|\sigma_n| = |\sigma_{7-n}|$
- $|C_k O_\ell| = |C_{4-k} O_{3-\ell}|$

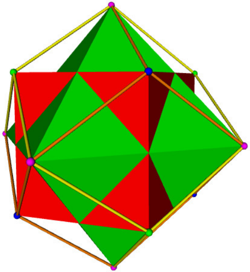
C_4O_0



C_0O_3

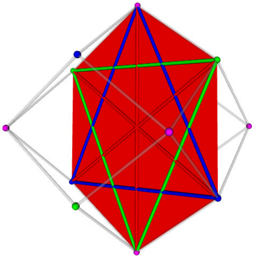


C_4O_3



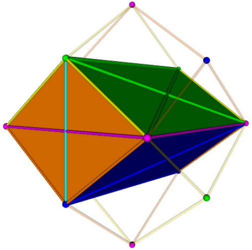
C_2O_1a

strong JSB hexagon



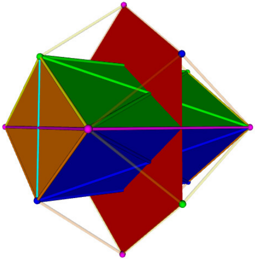
C_2O_2a

Buridan octagon



C_4O_3

rhombic dodecahedron



rhombicube

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structure	subtype	N	subtype	structure
σ_0	C_0O_0	1	C_4O_3	σ_7
σ_1	C_1O_0	4	C_3O_3	σ_6
	C_0O_1	3	C_4O_2	
	C_0O_2	3	C_4O_1	
σ_2	C_2O_0	6	C_2O_3	σ_5
	C_1O_1	12	C_3O_2	
	C_0O_3	1	C_4O_0	
	C_3O_0	4	C_1O_3	
σ_3	C_2O_1a	6	C_2O_2a	σ_4
	C_2O_1b	12	C_2O_2b	
	C_1O_2	12	C_3O_1	

- $\rightsquigarrow$ The logical geometry of rhombic dodecahedron RDH
- $\rightsquigarrow$ Typology of Aristotelian subdiagrams of RDH
- $\rightsquigarrow$ Tools/techniques for exhaustive analysis of internal structure of RDH
  - define σ_n-structure $= n$ out of the 7 PCDs of RDH
  - distinguish families of substructures $= C_k O_\ell$-perspective:
    $\sigma_n = [k$ out of the 4 PCDs of $C] + [\ell$ out of the 3 PCDs of $O]$
  - establish the exhaustiveness of the typology $\rightsquigarrow$ complementarity
- $\rightsquigarrow$ Frame of reference for classifying Aristotelian diagrams in the literature

σ_1	C_1O_0	Brown 1984
	C_0O_1	Demey 2012
	C_0O_2	Brown 1984, Béziau 2012
σ_2	C_2O_0	Fitting & Mendelsohn 1998, McNamara 2010, Lenzen 2012
	C_1O_1	Luzeaux, Sallantin & Dartnell 2008, Moretti 2009
	C_0O_3	Moretti 2009
	C_2O_1a	Sesmat 1951, Blanché 1966, Béziau 2012, Dubois & Prade 2013
σ_3	C_2O_1b	Czezowski 1955, Khomskii 2012, Chatti & Schang 2013
	C_1O_2	Seuren 2013, Seuren & Jaspers 2014, Smessaert & Demey 2014
	C_3O_0	Pellissier 2008, Moretti 2009, Moretti 2012
	C_1O_3	
	C_3O_1	
σ_4	C_2O_2b	Béziau 2003, Smessaert & Demey 2014
	C_2O_2a	Hughes 1987, Read 2012, Seuren 2012
	C_4O_0	Moretti 2009, Chatti & Schang 2013, Dubois & Prade 2013
	C_3O_2	Seuren & Jaspers 2014
σ_5	C_2O_3	
	C_4O_1	Blanché 1966, Joerden & Hruschka 1987, Wessels 2002
σ_6	C_4O_2	Béziau 2003, Moretti 2009, Moretti 2010
	C_3O_3	
σ_7	C_4O_3	Sauriol 1968, Moretti 2009, Smessaert 2009, Dubois & Prade 2013

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- high-level overview of logical geometry
- tension between two considerations:
  - the square is just one of many Aristotelian diagrams (typology)
  - the square is special after all (very informative)
- tension between symmetry and asymmetry $\rightsquigarrow$ work on lexicalisation patterns by Dany Jaspers (and Pieter Seuren)
- ongoing work:
  - concrete case studies: *many/few* ("filling in the gaps in the classification")
  - alternative presentations for Aristotelian diagrams (Square 2014)
  - relation between Aristotelian diagrams and other logic diagrams
    - duality diagrams (Diagrams 2012)
    - Hasse diagrams (Diagrams 2014)
  - graded Aristotelian relations

Thank you!

More info: www.logicalgeometry.org