Logical Geometries and Information in the Square of Oppositions

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This talk is based on joint work with Hans Smessaert.

- terminology: the *Aristotelian square* (historically inaccurate)
- visual representation of a fragment of the *Aristotelian geometry*
- geometry = formulas and relations between them

  (diagrams visualize modulo logical equivalence)

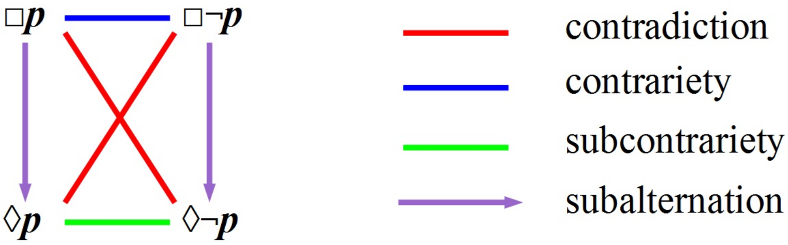
- the four Aristotelian relations (relative to a logical system S):

  φ and ψ are said to be

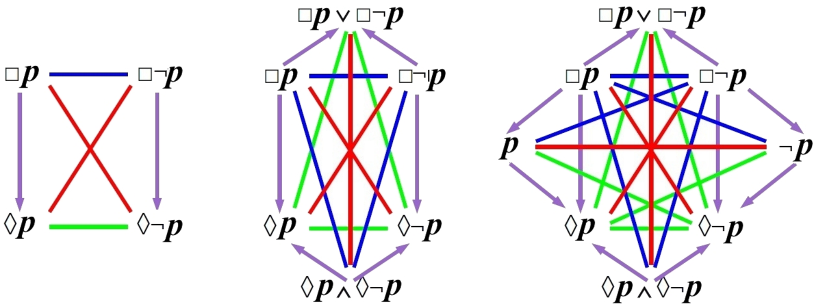
contradictory	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
in subalternation	iff	$S \models \varphi \to \psi$	and	$S \not\models \psi \to \varphi$

  (assumption: S has classical negation, conjunction, implication)

Our running example today: the modal logic S5.



- throughout history: several proposals to extend the square
  - more formulas, more relations
  - larger and more complex diagrams
  - hexagons, octagons, cubes and other three-dimensional figures...



- the square and its extensions: hexagon, octagon, RDH, ...
- the extensions are very interesting
  - well-motivated (singular propositions, Boolean closure)
  - throughout history (Sherwood hexagon, Buridan octagon)
  - interrelations (e.g. 3 squares inside JSB hexagon)

- yet:
  - (nearly) all logicians know about the square
  - (nearly) no logicians know about its extensions

- our explanation: "the Aristotelian square is very informative"
  - this claim sounds intuitive, but is also vague
  - provide precise and well-motivated framework

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- recall the Aristotelian geometry: φ and ψ are said to be

contradictory	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
in subalternation	iff	$S \models \varphi \rightarrow \psi$	and	$S \not\models \psi \rightarrow \varphi$

- problems with the Aristotelian geometry:
  - not mutually exclusive: e.g. $\bot$ and p are contrary *and* subaltern (problem disappears if we restrict to contingent formulas)
  - not exhaustive: e.g. p and $\Diamond p \wedge \Diamond\neg p$ are in no Arist. relation at all (if φ is contingent, then φ is in no Arist. relation to itself)
  - conceptual confusion: true/false together vs truth propagation
    - 'together' $\rightsquigarrow$ symmetrical relations (undirected)
    - 'propagation' $\rightsquigarrow$ asymmetrical relations (directed)

- the opposition geometry: φ and ψ are

contradictory	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
non-contradictory	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \not\models \neg(\neg\varphi \wedge \neg\psi)$

- the implication geometry: φ and ψ are in

bi-implication	iff	$S \models \varphi \rightarrow \psi$	and	$S \models \psi \rightarrow \varphi$
left-implication	iff	$S \models \varphi \rightarrow \psi$	and	$S \not\models \psi \rightarrow \varphi$
right-implication	iff	$S \not\models \varphi \rightarrow \psi$	and	$S \models \psi \rightarrow \varphi$
non-implication	iff	$S \not\models \varphi \rightarrow \psi$	and	$S \not\models \psi \rightarrow \varphi$

- opposition relations: being true/false together $\varphi \wedge \psi$ and $\neg\varphi \wedge \neg\psi$
- implication relations: truth propagation $\varphi \wedge \neg\psi$ and $\neg\varphi \wedge \psi$

- OG and IG jointly solve the problems of the Aristotelian geometry:
  - each pair of formulas stands in exactly one opposition relation
  - each pair of formulas stands in exactly one implication relation
  - no more conceptual confusion
- conceptual independence, yet clear relationship (symmetry breaking):

  CD(φ, ψ) $\Leftrightarrow$ BI$(\psi, \neg\varphi)$
  C(φ, ψ) $\Leftrightarrow$ LI$(\psi, \neg\varphi)$
  SC(φ, ψ) $\Leftrightarrow$ RI$(\psi, \neg\varphi)$
  NCD(φ, ψ) $\Leftrightarrow$ NI$(\psi, \neg\varphi)$
- Correia: two philosophical traditions
  - square as a theory of negation — commentaries on *De Interpretatione*
  - square as a theory of consequence — commentaries on *Prior Analytics*
- connection with binary connectives:
  - for $R \in$ OG and $S \in$ IG, we define a binary connective $\circ^{(R,S)}$
  - theorem: if $R(\varphi, \psi)$ and $S(\varphi, \psi)$, then $\models \varphi \circ^{(R,S)} \psi$

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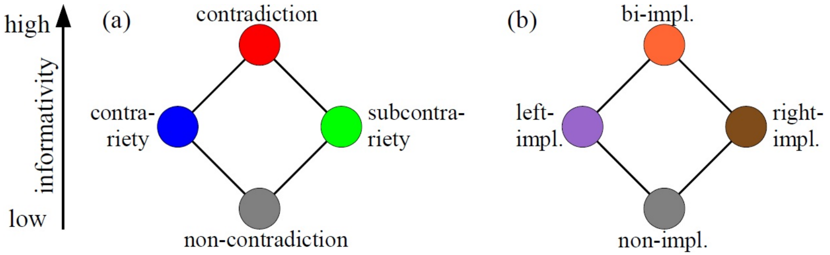
- informativity of a relation holding between φ and ψ is inversely correlated with the number of states (models) it is compatible with
- informativity of the opposition and implication relations:



- close match between formal account and intuitions:
  - e.g. CD is more informative than C
  - if φ is known,
    - announcing CD(φ, ψ) uniquely determines ψ
    - announcing C(φ, ψ) doesn't uniquely determine ψ

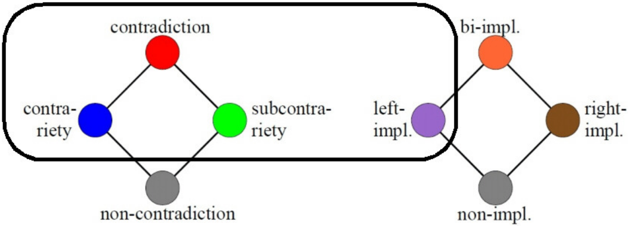
- combinatorial results on finite Boolean algebras
  - Boolean algebra $\mathbb{B}_n$ with 2^n formulas, formula of level i:
    - 1 contradictory
    - $2^{n-i} - 1$ contraries and $2^i - 1$ subcontraries
    - $(2^{n-i} - 1)(2^i - 1)$ non-contradictories
  - $1 < 2^{n-i} - 1, 2^i - 1 < (2^{n-i} - 1)(2^i - 1)$ iff $1 < i < n - 1$

- coherent with earlier results:
  - opposition and implication yield isomorphic informativity lattices
  - CD(φ, ψ) $\Leftrightarrow$ BI($\psi, \neg\varphi$), ...

- why is the Aristotelian square special?
- our answer: because it is very informative
  - it is a very informative *diagram*
  - in a very informative *geometry*

- Aristotelian geometry: hybrid between
  - opposition geometry: contradiction, contrariety, subcontrariety
  - implication geometry: left-implication (subalternation)

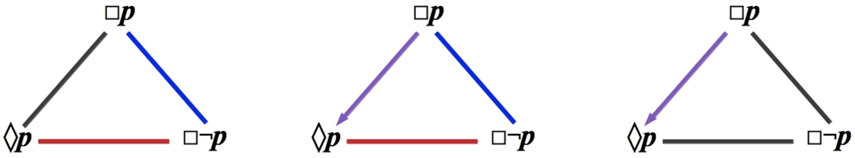
- these relations are highly informative (in their geometries)



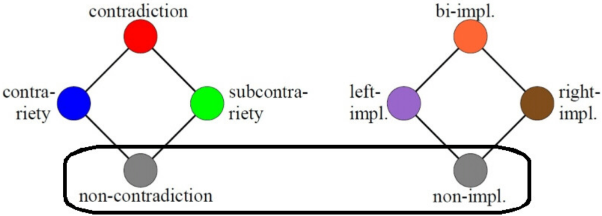
- given any two formulas:
  - they stand in exactly one opposition relation R
  - they stand in exactly one implication relation S

- if R is strictly more informative than S, then R is Aristotelian
- if S is strictly more informative than R, then S is Aristotelian

  - example 1: $\Box p$ and $\Diamond p$: non-contradiction and **left-implication**
  - example 2: $\Box p$ and $\Box\neg p$: **contrariety** and non-implication
  - example 3: $\Diamond p$ and $\Box\neg p$: **contradiction** and non-implication

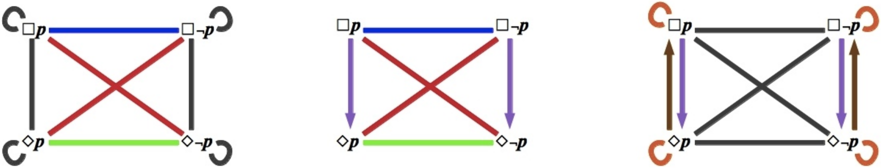


- given any two formulas:
  - they stand in exactly one opposition relation R
  - they stand in exactly one implication relation S
- what if neither relation is strictly more informative than the other?
- theorem: this can only occur in one case: NCD + NI (unconnectedness)

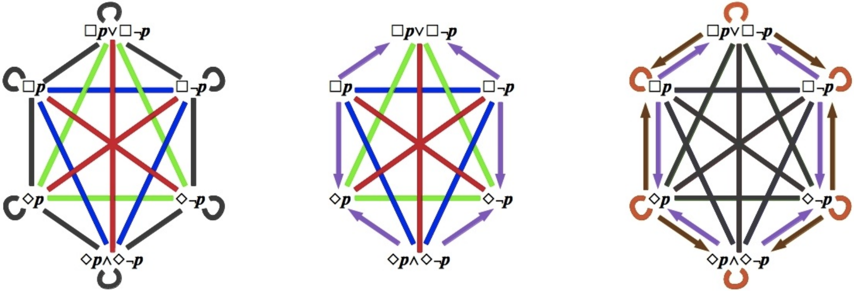


- Aristotelian gap = information gap
  - no Aristotelian relation at all (non-exhaustiveness of AG)
  - combination of the two *least informative* relations

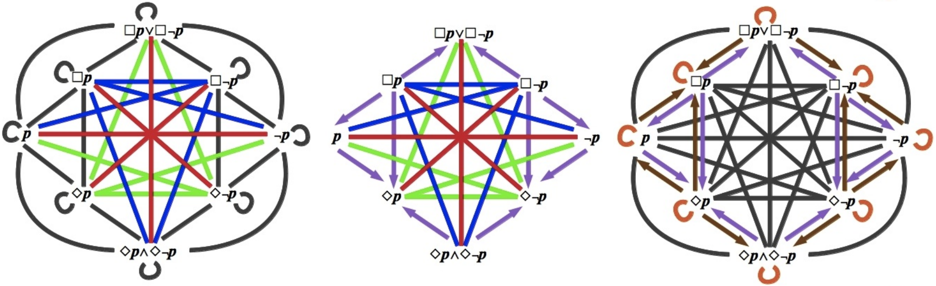
- no unconnectedness in the classical Aristotelian square



- no unconnectedness in the Jacoby-Sesmat-Blanché hexagon



- unconnectedness in the Béziau octagon
- e.g. p and $\Diamond p \wedge \Diamond \neg p$ are unconnected



- logical geometry: Aristotelian square of oppositions and its extensions
- the Aristotelian square is highly informative:
  - Aristotelian geometry is hybrid: *maximize* informativity
    $\Rightarrow$ applies to *all* Aristotelian diagrams
  - avoid unconnectedness: *minimize* uninformativity
    $\Rightarrow$ some Aristotelian diagrams succeed better than others
    - classical square, JSB hexagon, SC hexagon don't have unconnectedness
    - Béziau octagon (and many other diagrams) do have unconnectedness
- Q: what about the JSB hexagon, SC hexagon, etc.?
  - equally informative as the square
  - yet less widely known. . .
- A: requires yet another geometry: duality

Thank you!

More info: www.logicalgeometry.org