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# Ordering relations, partitions and Aristotelian diagrams

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1. Introduction
2. The categorical statements from syllogistics
3. Propositional logic
4. Total ordering relations
5. Partial ordering relations
6. Total ordering relations, once again
7. Conclusion

1. **Introduction**
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- scalarity in mathematics: **ordering relations**
- partial ordering $\leq$ on a set $D$:
  - reflexivity: $\forall x \in D : x \leq x$
  - transitivity: $\forall x, y, z \in D : x \leq y, y \leq z \Rightarrow x \leq z$
  - antisymmetry: $\forall x, y \in D : x \leq y, y \leq x \Rightarrow x = y$
- total ordering $\leq$ on a set $D$:
  - all the properties of partial orderings
  - totality: $\forall x, y \in D : x \leq y$ or $y \leq x$
- today: the role of ordering relations in **logical geometry**

- systematic study of the well-known **Aristotelian relations**:
  two statements are said to be

| | | |
|---|---|---|
| *contradictory* | iff | they cannot be true together and they cannot be false together |
| *contrary* | iff | they cannot be true together but they can be false together |
| *subcontrary* | iff | they can be true together but they cannot be false together |
| *in subalternation* | iff | the first proposition entails the second but the second doesn't entail the first |

- an **Aristotelian diagram** is a visual representation of
  - a fragment $\mathcal{F}$ of formulas (/natural language expressions/...)
  - the Aristotelian relations holding between those formulas

- consider a fragment of formulas $\mathcal{F}$
- the **partition** of logical space that is induced by $\mathcal{F}$ is
  $\Pi(\mathcal{F}) := \{\alpha \in \mathcal{L} \mid \alpha \equiv \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m$, and $\alpha$ is consistent$\}$
- the elements of $\Pi(\mathcal{F})$ are called **anchor formulas**
- ordering relations/scalarity phenomena can play a role in the fragment $\mathcal{F}$ as well as in the partition $\Pi(\mathcal{F})$
- **diagrammatic** representation:

| | | | |
|---|---|---|---|
| logical realm | fragment $\mathcal{F}$ | $\xrightarrow{\text{induces}}$ | partition $\Pi(\mathcal{F})$ |
| $\downarrow$ | $\downarrow$ | | $\downarrow$ |
| visual realm | Aristotelian diagram | | partition diagram |

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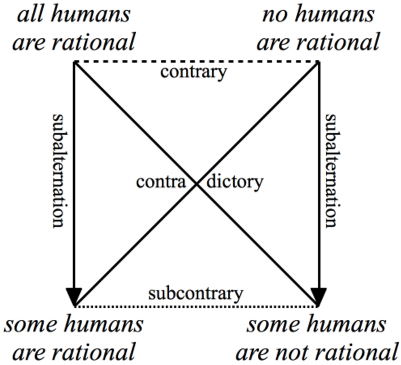
- consider the fragment of the four **categorical statements**:

$$\mathcal{F}_c \quad := \quad \{ \quad \begin{array}{l} \textit{all humans are rational}, \\ \textit{some humans are rational}, \\ \textit{no humans are rational}, \\ \textit{some humans are not rational} \quad \} \end{array}$$

- note: $\mathcal{F}_c$ does **not** seem to exhibit any **ordering** relation

- fragment $\mathcal{F}_c$ of the four categorical statements
- Aristotelian diagram for $\mathcal{F}_c$: **classical square of opposition** (under the assumption of existential import)



- fragment $\mathcal{F}_c$ of the four categorical statements
- the partition induced by $\mathcal{F}_c$:

$$\Pi(\mathcal{F}_c) = \{ \begin{array}{l} \textit{all humans are rational}, \\ \textit{some but not all humans are rational}, \\ \textit{no humans are rational} \end{array} \}$$

- (the size of) the partition $\Pi(\mathcal{F}_c)$ allows us to measure the Boolean complexity of the fragment $\mathcal{F}_c$
    - $|\Pi(\mathcal{F}_c)| = 3$
    - the Boolean closure of $\mathcal{F}_c$ contains $2^3 = 8$ formulas
    - up to logical equivalence, there are $8$ Boolean combinations of $\mathcal{F}_c$-formulas

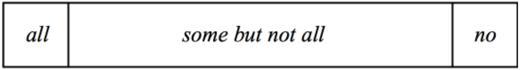
## The categorical statements from syllogistics

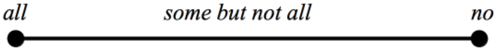


- the partition induced by $\mathcal{F}_c$:

$$\Pi(\mathcal{F}_c) = \{ \textit{all humans are rational}, \textit{some but not all humans are rational}, \textit{no humans are rational} \}$$

- diagrammatic representations of $\Pi(\mathcal{F}_c)$:




- note: $\Pi(\mathcal{F}_c)$ constitutes a **total ordering** of logical space

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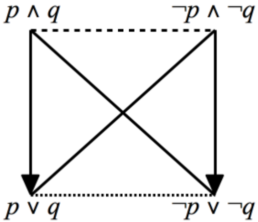
- consider the fragment $\mathcal{F}_1$, which contains four formulas from **propositional logic**:

$$\mathcal{F}_1 \quad := \quad \{ \quad \begin{array}{l} p \wedge q, \\ p \vee q, \\ \neg p \wedge \neg q, \\ \neg p \vee \neg q \end{array} \quad \}$$

- note: $\mathcal{F}_1$ does **not** exhibit any **ordering** relation

- fragment $\mathcal{F}_1$ of four formulas from propositional logic
- Aristotelian diagram for $\mathcal{F}_1$: **classical square of opposition**





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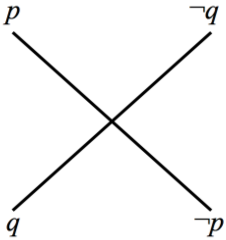
- consider the fragment $\mathcal{F}_2$, which again contains four formulas from **propositional logic**:

$$\mathcal{F}_2 \quad := \quad \{ \quad \begin{array}{l} p, \\ q, \\ \neg p, \\ \neg q \quad \} \end{array}$$

- note: $\mathcal{F}_2$ does **not** exhibit any **ordering** relation

- fragment $\mathcal{F}_2$ of four formulas from propositional logic
- Aristotelian diagram for $\mathcal{F}_2$: **degenerate square of opposition**
  - contradictions between $p/\neg p$ and $q/\neg q$
  - all other pairs of formulas are unconnected: they do not stand in any Aristotelian relation at all

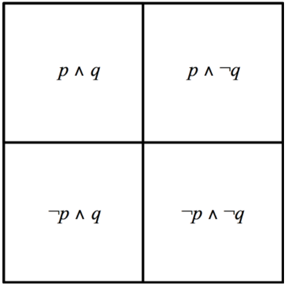


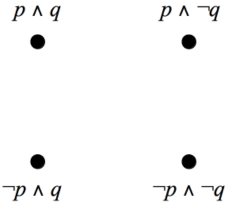
- the partition induced by $\mathcal{F}_2$:

$$\Pi(\mathcal{F}_2) = \{ \begin{array}{l} p \wedge q, \\ p \wedge \neg q, \\ \neg p \wedge q, \\ \neg p \wedge \neg q \end{array} \}$$

- (the size of) the partition $\Pi(\mathcal{F}_2)$ allows us to measure the Boolean complexity of the fragment $\mathcal{F}_2$
  - $|\Pi(\mathcal{F}_2)| = 4$
  - the Boolean closure of $\mathcal{F}_2$ contains $2^4 = 16$ formulas
  - up to logical equivalence, there are $16$ Boolean combinations of $\mathcal{F}_2$-formulas

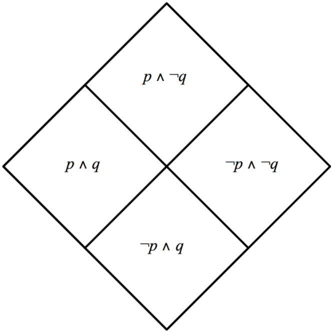
- diagrammatic representations of $\Pi(\mathcal{F}_2)$:

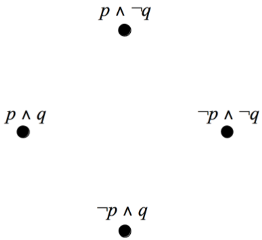



- note: $\Pi(\mathcal{F}_2)$ does **not** involve any underlying **ordering** of logical space
  - $\Pi(\mathcal{F}_2)$ displays a high degree of **symmetry**
  - $\Pi(\mathcal{F}_2)$ is the result of crosscutting the two bipartitions $p/\neg p$ and $q/\neg q$

- one might argue that $\Pi(\mathcal{F}_2)$ is an ordering of logical space after all:
  - not a total ordering, but a partial ordering
  - anchor formulas are ordered by 'number of true (non-negated) conjuncts'




- however, in most concrete cases, this does not seem very plausible
  e.g. the crosscutt of the bipartitions *male/female* and *adult/child*

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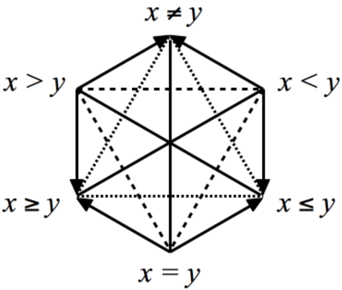
- consider the fragment $\mathcal{F}_t$ of six statements involving a **total ordering relation** $\leq$ on a set $D$ and two elements $x, y \in D$:

$$\mathcal{F}_t \quad := \quad \{ \quad \begin{array}{l} x > y, \\ x = y, \\ x < y, \\ x \leq y, \\ x \neq y, \\ x \geq y \end{array} \quad \}$$

- Aristotelian diagram for $\mathcal{F}_t$: a **hexagon of opposition**




- originally due to Robert Blanché (*Sur l'opposition des concepts*, 1953)

- the partition induced by $\mathcal{F}_t$:

$$\Pi(\mathcal{F}_t) \quad = \quad \{ \quad x > y, \\ x = y, \\ x < y \quad \}$$

- (the size of) the partition $\Pi(\mathcal{F}_t)$ allows us to measure the Boolean complexity of the fragment $\mathcal{F}_t$
    - $|\Pi(\mathcal{F}_t)| = 3$
    - the Boolean closure of $\mathcal{F}_t$ contains $2^3 = 8$ formulas
    - up to logical equivalence, there are $8$ Boolean combinations of $\mathcal{F}_t$-formulas
    - apart from $\perp$ and $\top$, all of these Boolean combinations can already be found in the hexagon itself
    - the hexagon is **closed under the Boolean operations**

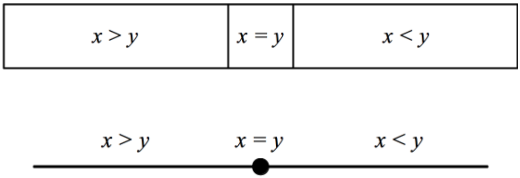
## Total ordering relations



- the partition induced by $\mathcal{F}_t$:

$$\Pi(\mathcal{F}_t) \quad = \quad \{ \quad x > y, \quad x = y, \quad x < y \quad \}$$

- diagrammatic representations of $\Pi(\mathcal{F}_t)$:




- note: $\Pi(\mathcal{F}_t)$ constitutes itself a **total ordering** of logical space

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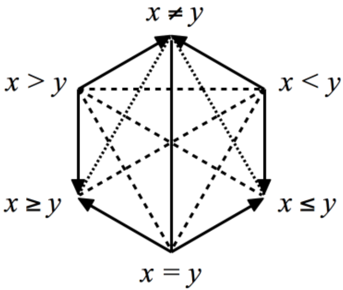
- let $\mathcal{F}_p$ be exactly the same fragment as before ($\mathcal{F}_t$), but now under the assumption that $\leq$ is a **partial ordering** on $D$ instead of a total ordering

$$\mathcal{F}_p \quad := \quad \{ \quad \begin{array}{l} x > y, \\ x = y, \\ x < y, \\ x \leq y, \\ x \neq y, \\ x \geq y \end{array} \quad \}$$

- we drop the assumption of **totality** ($\forall x, y \in D : x \leq y$ or $y \leq x$)

- it becomes possible for $x$ and $y$ to be **incomparable**: $x \# y$ (i.e. neither $x \geq y$ nor $x \leq y$)

- the Aristotelian diagram for $\mathcal{F}_p$:
  a very different **hexagon of opposition**
  - two of the three contradictions change into contrarieties ($>$ / $\leq$ and $<$ / $\geq$)
  - one of the three subcontrarieties is lost ($\geq$ / $\leq$)
  - the three contrarieties and six subalternations remain unchanged



- the partition induced by $\mathcal{F}_p$:

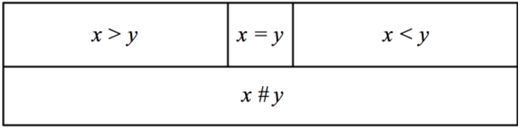
$$\Pi(\mathcal{F}_p) \quad = \quad \{ \quad \begin{array}{l} x > y, \\ x = y, \\ x < y, \\ x \# y \quad \} \end{array}$$

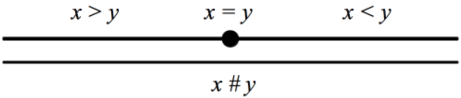
- (the size of) the partition $\Pi(\mathcal{F}_p)$ allows us to measure the Boolean complexity of the fragment $\mathcal{F}_p$
  - $|\Pi(\mathcal{F}_t)| = 4$
  - the Boolean closure of $\mathcal{F}_p$ contains $2^4 = 16$ formulas
  - up to logical equivalence, there are $16$ Boolean combinations of $\mathcal{F}_p$-formulas

# Partial ordering relations



- diagrammatic representations of $\Pi(\mathcal{F}_p)$:




- note: $\Pi(\mathcal{F}_p)$ constitutes itself a **partial ordering** of logical space

- by setting $\#$ to be $\emptyset$
  (i.e. imposing the requirement that $x \# y$ is impossible):
  - from partial ordering to total ordering
  - from the 4-partition $\Pi(\mathcal{F}_p)$ to 3-partition $\Pi(\mathcal{F}_t)$
  - from Boolean closure of size $2^4 = 16$ to Boolean closure of size $2^3 = 8$

| partial ordering | | total ordering | total ordering | | partial ordering |
|---|---|---|---|---|---|
| $>$ | $\rightarrow$ | $>$ | $= \cup <$ | $\leftarrow$ | $= \cup < \cup \#$ |
| $> \cup \#$ | $\nearrow$ | | | $\nwarrow$ | $= \cup <$ |
| $=$ | $\rightarrow$ | $=$ | $> \cup <$ | $\leftarrow$ | $> \cup < \cup \#$ |
| $= \cup \#$ | $\nearrow$ | | | $\nwarrow$ | $> \cup <$ |
| $<$ | $\rightarrow$ | $<$ | $> \cup =$ | $\leftarrow$ | $> \cup = \cup \#$ |
| $< \cup \#$ | $\nearrow$ | | | $\nwarrow$ | $> \cup =$ |
| $\#$ | $\rightarrow$ | $\emptyset$ | $> \cup = \cup <$ | $\leftarrow$ | $> \cup = \cup <$ |
| $\emptyset$ | $\nearrow$ | | | $\nwarrow$ | $> \cup = \cup < \cup \#$ |

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- so far:
  - focus on total ordering versus partial ordering
  - focus on the axiom of **totality**

- now:
  - focus on the axiom of **transitivity**
  - $\forall x, y, z \in D : x \leq y, y \leq z \Rightarrow x \leq z$

- consider the fragment $\mathcal{F}^*$, which, for three elements $x, y, z \in D$, contains all formulas of the form $x \circ y$, $y \circ z$ and $x \circ z$, with $\circ \in \{>, =, <, \leq, \neq, \geq\}$

- note: $|\mathcal{F}^*| = 3 \times 6 = 18$

- what is the partition $\Pi(\mathcal{F}^*)$ that is induced by $\mathcal{F}^*$?

# Total ordering relations, once again

- we can write $\mathcal{F}^* = \mathcal{F}_{xy} \cup \mathcal{F}_{yz} \cup \mathcal{F}_{xz}$
  - $\mathcal{F}_{xy} = \{x > y, x = y, x < y, x \leq y, x \neq y, x \geq y\}$ (Blanché hexagon)
  - $\mathcal{F}_{yz} = \{y > z, y = z, y < z, y \leq z, y \neq z, y \geq z\}$ (Blanché hexagon)
  - $\mathcal{F}_{xz} = \{x > z, x = z, x < z, x \leq z, x \neq z, x \geq z\}$ (Blanché hexagon)
- we know the partitions that are induced by these subfragments of $\mathcal{F}^*$:
  - $\Pi(\mathcal{F}_{xy}) = \{x > y, x = y, x < y\}$
  - $\Pi(\mathcal{F}_{yz}) = \{y > z, y = z, y < z\}$
  - $\Pi(\mathcal{F}_{xz}) = \{x > z, x = z, x < z\}$
- $\Pi(\mathcal{F}^*)$ is the result of crosscutting $\Pi(\mathcal{F}_{xy})$, $\Pi(\mathcal{F}_{yz})$ and $\Pi(\mathcal{F}_{xz})$
  - in principle $3 \times 3 \times 3 = 27$ conjunctions of anchor formulas
  - because of **transitivity**, many of these conjunctions are inconsistent (e.g. $x > y$, $y > z$, and $x < z$ are inconsistent with each other)
  - exactly 13 conjunctions are consistent, and thus get included in $\Pi(\mathcal{F}^*)$

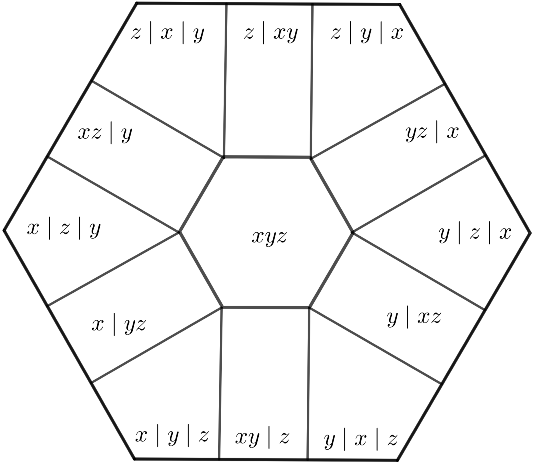
- the partition $\Pi(\mathcal{F}^*)$ contains the following 13 formulas:

| | | |
|---|---|---|
| 1. | $x > y \quad \wedge \quad y > z \quad \wedge \quad x > z$ | $x \mid y \mid z$ |
| 2. | $x = y \quad \wedge \quad y > z \quad \wedge \quad x > z$ | $xy \mid z$ |
| 3. | $x < y \quad \wedge \quad y > z \quad \wedge \quad x > z$ | $y \mid x \mid z$ |
| 4. | $x > y \quad \wedge \quad y = z \quad \wedge \quad x > z$ | $x \mid yz$ |
| 5. | $x > y \quad \wedge \quad y < z \quad \wedge \quad x > z$ | $x \mid z \mid y$ |
| 6. | $x < y \quad \wedge \quad y > z \quad \wedge \quad x = z$ | $y \mid xz$ |
| 7. | $x = y \quad \wedge \quad y = z \quad \wedge \quad x = z$ | $xyz$ |
| 8. | $x > y \quad \wedge \quad y < z \quad \wedge \quad x = z$ | $xz \mid y$ |
| 9. | $x < y \quad \wedge \quad y > z \quad \wedge \quad x < z$ | $y \mid z \mid x$ |
| 10. | $x < y \quad \wedge \quad y = z \quad \wedge \quad x < z$ | $yz \mid x$ |
| 11. | $x > y \quad \wedge \quad y < z \quad \wedge \quad x < z$ | $z \mid x \mid y$ |
| 12. | $x = y \quad \wedge \quad y < z \quad \wedge \quad x < z$ | $z \mid xy$ |
| 13. | $x < y \quad \wedge \quad y < z \quad \wedge \quad x < z$ | $z \mid y \mid x$ |

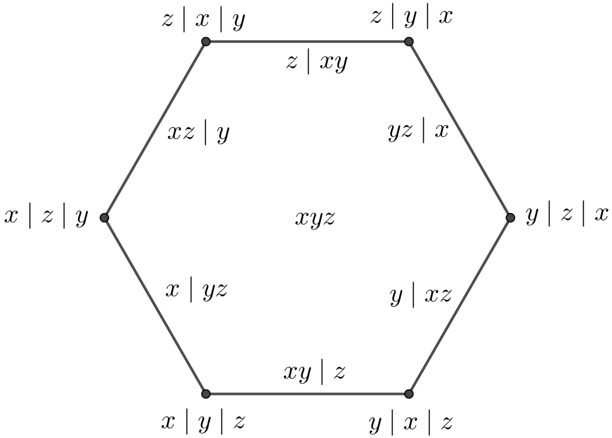
- (the size of) the partition $\Pi(\mathcal{F}^*)$ allows us to measure the Boolean complexity of the fragment $\mathcal{F}^*$
  - recall that $|\mathcal{F}^*| = 18$
  - we have just seen that $|\Pi(\mathcal{F}^*)| = 13$
  - the Boolean closure of $\mathcal{F}^*$ contains $2^{13} = 8.192$ formulas
  - up to logical equivalence, there are $8.192$ Boolean combinations of $\mathcal{F}^*$-formulas

- the partition $\Pi(\mathcal{F}^*)$ is **not** an **ordering** on logical space, but rather has a high degree of **symmetry**
  - 6 conjunctions with 0 identity-conjuncts
  - 6 conjunctions with 1 identity-conjunct
  - 1 conjunction with 3 identity-conjuncts

- a diagrammatic representation of $\Pi(\mathcal{F}^*)$



- another diagrammatic representation of $\Pi(\mathcal{F}^*)$ (geometric combinatorics: **permutahedron**)



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# Conclusion

$$\begin{array}{ccccc} \text{logical realm} & \text{fragment } \mathcal{F} & \xrightarrow{\text{induces}} & \text{partition } \Pi(\mathcal{F}) \\ \downarrow & \downarrow & & \downarrow \\ \text{visual realm} & \text{Aristotelian diagram} & & \text{partition diagram} \end{array}$$

ordering relations/scalarity phenomena can play a role
in the fragment $\mathcal{F}$ as well as in the partition $\Pi(\mathcal{F})$

| fragment/ Aristotelian diagram | partition/ partition diagram | concrete example |
|---|---|---|
| not order-based | order-based | cf. section 2: $\mathcal{F}_c$ |
| not order-based | not order-based | cf. section 3: $\mathcal{F}_2$ |
| order-based | order-based | cf. sections 4,5: $\mathcal{F}_t$, $\mathcal{F}_p$ |
| order-based | not order-based | cf. section 6: $\mathcal{F}^*$ |

# Thank you!

# Questions?

More info: `www.logicalgeometry.org`



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