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Buridan's and Avicenna's Aristotelian Diagrams for Combined Operators

Lorenz Demey

CLAW/DWMC Symposium, 22 May '15

- this talk is based on joint research with
  - Saloua Chatti (U. Tunis)
  - Hans Smessaert (KU Leuven)
  - Fabien Schang (HSE Moscow)

- mix of logical and historical aspects
- today's talk:
  - first half: emphasis on the historical aspects
  - second half: emphasis on the more technical aspects

- historical scholarship:
  - Buridan: S. Read, G. Hughes, S. Johnston, J. Campos Benítez
  - Avicenna: S. Chatti, W. Hodges

- status of diagrams:
  - heavyweight: **visual representation** of logical theory
  - lightweight: visual representation of **logical theory**

Goals of the talk

- Buridan's Aristotelian octagons:
  - relatively well-known
  - actual diagrams
- logical goals:
  - systematically study some natural extensions of Buridan's octagon
  - compare them in terms of their logical complexity (bitstring length)
- historical goals:
  - show that although he did not draw the actual diagram, Buridan had the logical means available to construct at least one of these extensions
  - establish the historical priority of Al-Farabi and Avicenna with respect to Buridan's octagon and at least two of its extensions

1 Some Preliminaries from Logical Geometry

2 Buridan's Aristotelian Diagrams

3 Avicenna's Aristotelian Diagrams

4 Bitstring Analysis

5 Conclusion

1. Some Preliminaries from Logical Geometry
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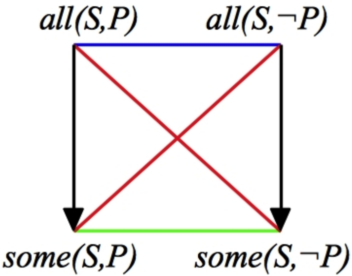
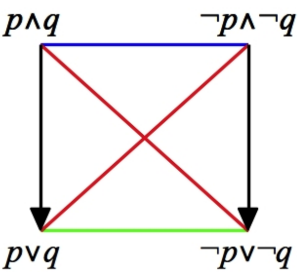
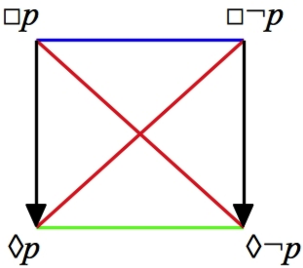
- an Aristotelian diagram visualizes some formulas and the Aristotelian relations holding between them
- definition of the Aristotelian relations: two propositions are

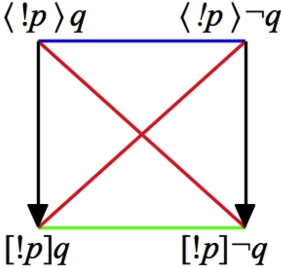
contradictory	iff	they cannot be true together and they cannot be false together,
contrary	iff	they cannot be true together but they can be false together,
subcontrary	iff	they can be true together but they cannot be false together,
in subalternation	iff	the first proposition entails the second but the second doesn't entail the first

Some Aristotelian squares

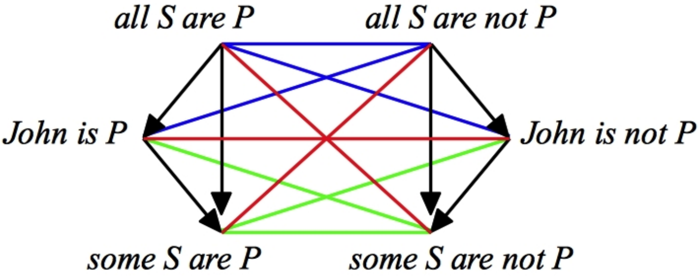








- already during the Middle Ages, philosophers used Aristotelian diagrams larger than the classical square to visualize their logical theories
- e.g. John Buridan (ca. 1295–1358): several octagons (see later)
- e.g. William of Sherwood (ca. 1200–1272), *Introductiones in Logicam* $\Rightarrow$ integrating singular propositions into the classical square



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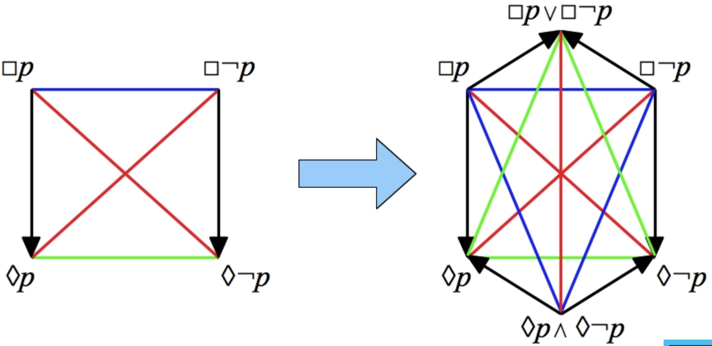
Boolean closure of an Aristotelian diagram

- the smallest Aristotelian diagram that contains all contingent Boolean combinations of formulas from the original diagram
- the Boolean closure of a classical square is a Jacoby-Sesmat-Blanché hexagon (6 formulas)



Boolean closure of an Aristotelian diagram

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- the Boolean closure of a classical square is a Jacoby-Sesmat-Blanché hexagon (6 formulas)
- the Boolean closure of a Sherwood-Czezowski hexagon is a (3D) rhombic dodecahedron (14 formulas)

Theorem

A Boolean closure has $2^n - 2$ formulas, for some natural number n.

- every Aristotelian diagram can be represented by means of bitstrings
- bitstring = sequence of bits (0/1)

- 'anchor formulas' $\alpha_1, \ldots, \alpha_n$ (obtainable from the diagram)
- every formula in (the Boolean closure of) the diagram is equivalent to a disjunction of these anchor formulas
- bitstrings keep track which anchor formulas occur in the disjunction and which ones do not

- technical: disjunctive normal forms
- intuition: bitstrings as coordinates, anchor formulas as axes

$$\text{point } (5,2) \leftrightsquigarrow 5 \cdot \vec{\mathbf{x}} + 2 \cdot \vec{\mathbf{y}}$$

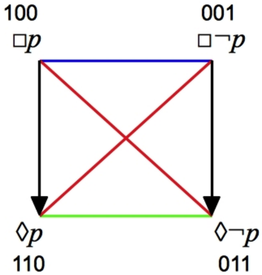
- bitstrings of length n $\Leftrightarrow$ size of Boolean closure is $2^n - 2$

- example: modal square $\Rightarrow$ bitstrings of length $n = 3$
- anchor formulas:

$\alpha_1 = \Box p$ e.g. $\Diamond p \equiv \Box p \vee (\Diamond p \wedge \Diamond \neg p) = \alpha_1 \vee \alpha_2 = 110$

$\alpha_2 = \Diamond p \wedge \Diamond \neg p$

$\alpha_3 = \Box \neg p$



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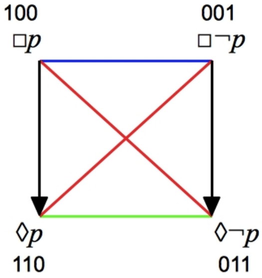

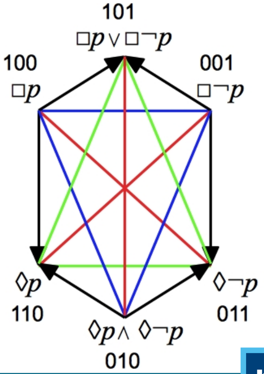

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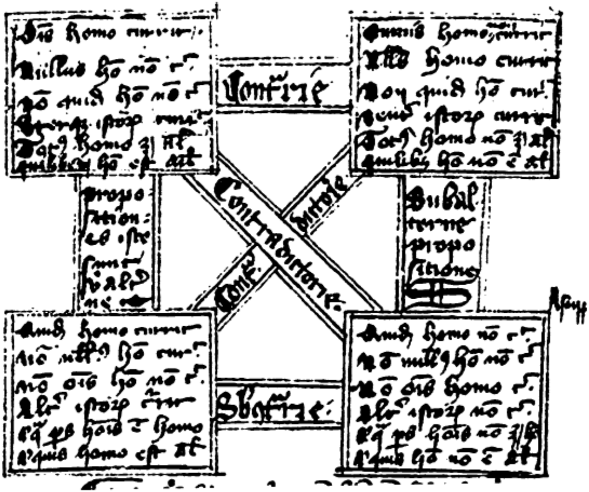
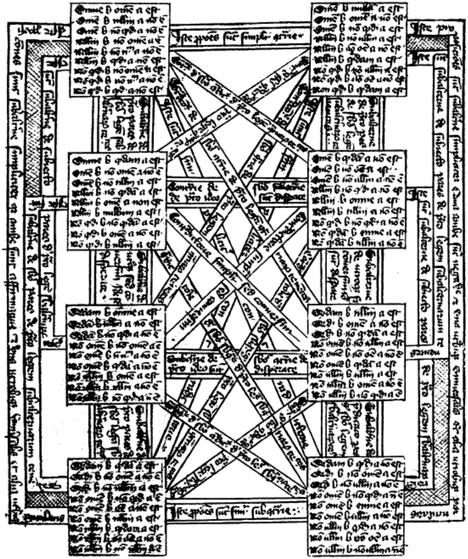
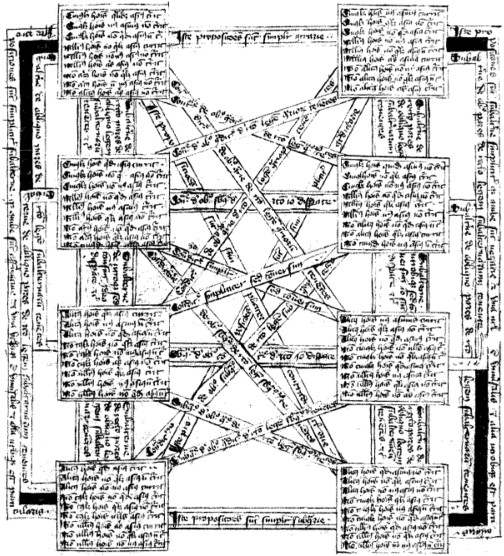
- John Buridan, ca. 1295–1358
- *Summulae de Dialectica* (late 1330s, revisions into the 1350s)
- Vatican manuscript Pal.Lat. 994 contains several Aristotelian diagrams:
    - Aristotelian square for the usual categorical propositions (A,I,E,O) (e.g. "every human is mortal")
    - Aristotelian octagon for non-normal propositions (e.g. "every human some animal is not") (cf. regimentation of Latin)
    - Aristotelian octagon for propositions with oblique terms (e.g. "every donkey of every human is running")
    - Aristotelian octagon for modal propositions (e.g. "every human is necessarily mortal")

$$\begin{array}{rcl} \text{square} & \Rightarrow & \text{single operator} \\ \text{octagons} & \Rightarrow & \text{combined operators} \end{array}$$


Contradictorie



Buridan's octagon for propositions with oblique terms



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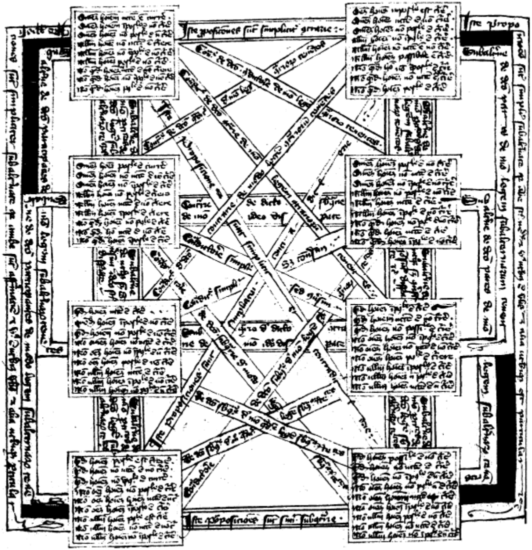

- Buridan's octagon contains the following 8 formulas:
  1. all A are necessarily B $\forall x(\Diamond Ax \to \Box Bx)$ $\forall\Box$
  2. all A are possibly B $\forall x(\Diamond Ax \to \Diamond Bx)$ $\forall\Diamond$
  3. some A are necessarily B $\exists x(\Diamond Ax \wedge \Box Bx)$ $\exists\Box$
  4. some A are possibly B $\exists x(\Diamond Ax \wedge \Diamond Bx)$ $\exists\Diamond$
  5. all A are necessarily not B $\forall x(\Diamond Ax \to \Box\neg Bx)$ $\forall\Box\neg$
  6. all A are possibly not B $\forall x(\Diamond Ax \to \Diamond\neg Bx)$ $\forall\Diamond\neg$
  7. some A are nessarily not B $\exists x(\Diamond Ax \wedge \Box\neg Bx)$ $\exists\Box\neg$
  8. some A are possibly not B $\exists x(\Diamond Ax \wedge \Diamond\neg Bx)$ $\exists\Diamond\neg$
- note: de re modality, ampliation of the subject in modal formulas
- watch out with negative formulas:

no A are necessarily B
= no A are (necessarily B)
= all A are not (necessarily B)
= all A are possibly not B

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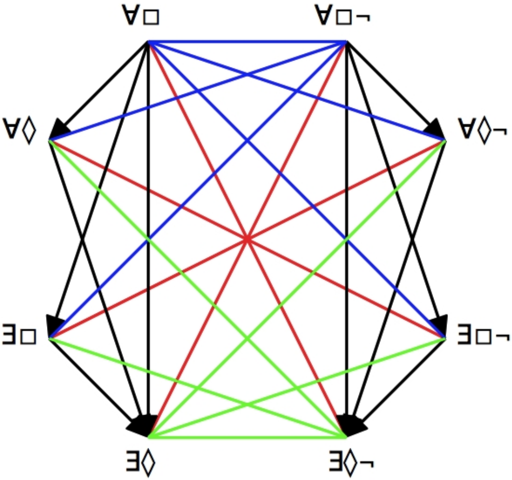
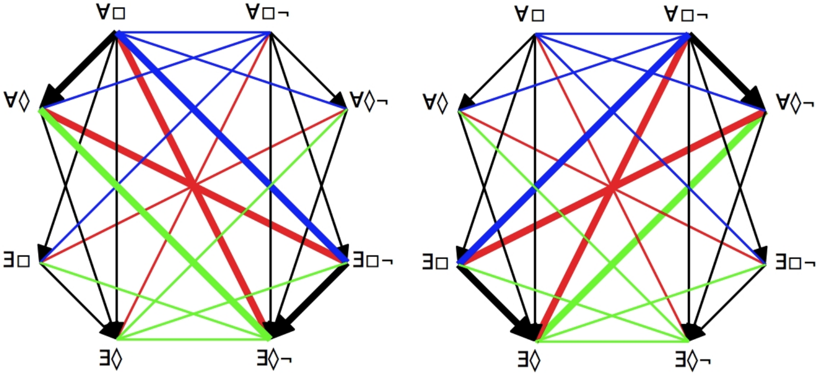
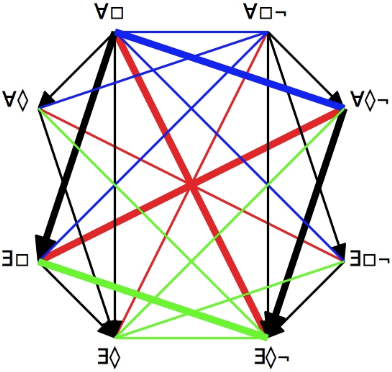
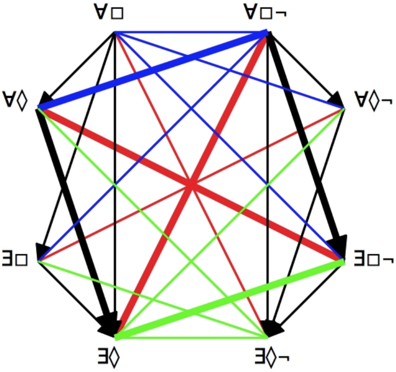




(assumption: $\Box\varphi \rightarrow \Diamond\varphi$)

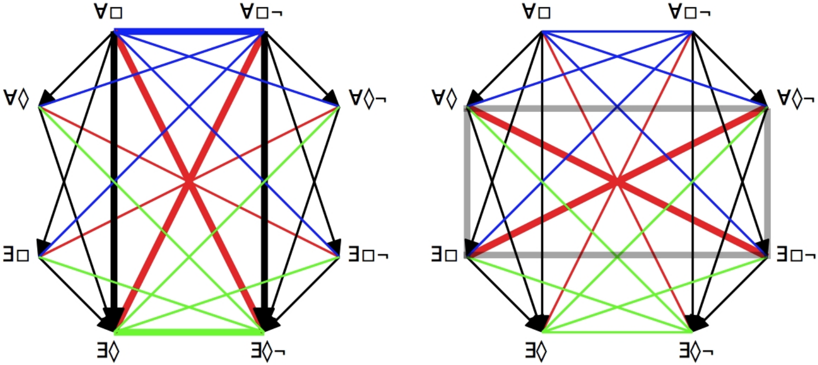
The internal structure of Buridan's modal octagon





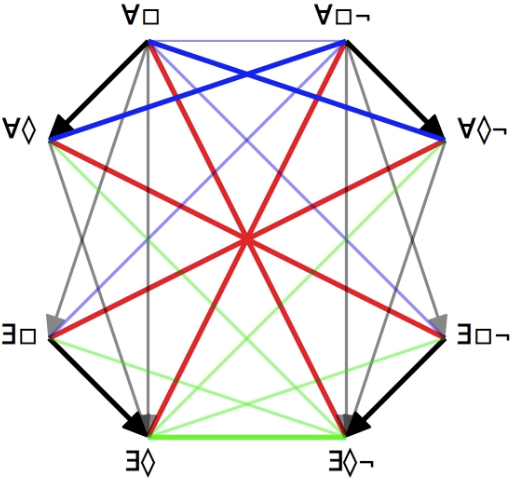
(assumption: $\exists x \Diamond Ax$ — amplified version of existential import!)

The internal structure of Buridan's modal octagon



(note: unconnectedness square in the middle of the octagon)

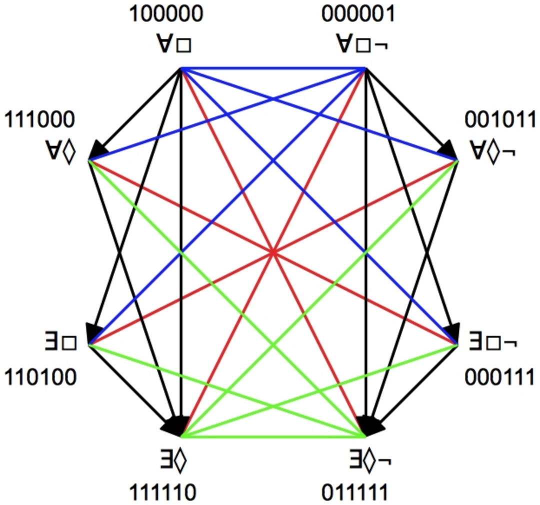
- S. Chatti, 2015, Al-Farabi on Modal Oppositions
- Al-Farabi: ca. 873–950 ($\pm$ 400 years before Buridan)
  - identified the 8 formulas of Buridan's octagon
  - identified some of the Aristotelian relations of the octagon (but all relations are deducible from the ones identified by Al-Farabi)
- unlike Buridan, Al-Farabi does not seem to have visualized his logical theorizing by means of an actual diagram
- unlike Buridan, Al-Farabi was not explicit about the issue of ampliation



- we can define a bitstring representation for Buridan's modal octagon
- this makes use of bitstrings of **length 6**

- 6 anchor formulas:
  1. $\forall\Box$
  2. $\forall\Diamond \wedge \exists\Box \wedge \exists\Diamond\neg$
  3. $\forall\Diamond \wedge \forall\Diamond\neg$
  4. $\exists\Box \wedge \exists\Box\neg$
  5. $\forall\Diamond\neg \wedge \exists\Box\neg \wedge \exists\Diamond$
  6. $\forall\Box\neg$

- note: this means that the Boolean closure of the octagon contains $2^6 - 2 = 62$ formulas



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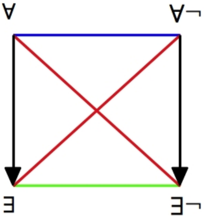
- classical square (representable by bitstrings of length 3)
  $\Rightarrow$ natural extension: JSB hexagon, i.e. its Boolean closure ($6 = 2^3 - 2$)

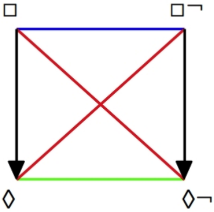
- Buridan's modal octagon (representable by bitstrings of length 6)
  $\Rightarrow$ its Boolean closure has $2^6 - 2 = 62$ formulas $\Rightarrow$ too large!
  $\Rightarrow$ other, more 'reasonable' extensions of the octagon?

- key idea:

Buridan's octagon for quantified modal logic can be seen as arising out of the interaction of a quantifier square and a modality square

instead of taking the Boolean closure of the entire octagon, we can take the Boolean closure of its 'component squares'





The quantifier square contains $\forall, \forall\neg, \exists, \exists\neg$.
The modality square contains $\Box, \Box\neg, \Diamond, \Diamond\neg$.

	$\Box$	$\Box\neg$	$\Diamond$	$\Diamond\neg$
$\forall$	$\forall\Box$	$\forall\Box\neg$	$\forall\Diamond$	$\forall\Diamond\neg$
$\forall\neg$	$\forall\neg\Box$	$\forall\neg\Box\neg$	$\forall\neg\Diamond$	$\forall\neg\Diamond\neg$
$\exists$	$\exists\Box$	$\exists\Box\neg$	$\exists\Diamond$	$\exists\Diamond\neg$
$\exists\neg$	$\exists\neg\Box$	$\exists\neg\Box\neg$	$\exists\neg\Diamond$	$\exists\neg\Diamond\neg$

- square $\times$ square $\Rightarrow 4 \times 4 = 16$ formulas

	$\Box$	$\Box\neg$	$\Diamond$	$\Diamond\neg$
$\forall$	$\forall\Box$	$\forall\Box\neg$	$\forall\Diamond$	$\forall\Diamond\neg$
$\forall\neg$	$\forall\neg\Box$	$\forall\neg\Box\neg$	$\forall\neg\Diamond$	$\forall\neg\Diamond\neg$
$\exists$	$\exists\Box$	$\exists\Box\neg$	$\exists\Diamond$	$\exists\Diamond\neg$
$\exists\neg$	$\exists\neg\Box$	$\exists\neg\Box\neg$	$\exists\neg\Diamond$	$\exists\neg\Diamond\neg$

- these 16 formulas are pairwise equivalent:

$$\begin{array}{lcl}
\forall\neg\Box & \equiv & \forall\Diamond\neg \\
\forall\neg\Box\neg & \equiv & \forall\Diamond \\
\forall\neg\Diamond & \equiv & \forall\Box\neg \\
\forall\neg\Diamond\neg & \equiv & \forall\Box \\
\exists\neg\Box & \equiv & \exists\Diamond\neg \\
\exists\neg\Box\neg & \equiv & \exists\Diamond \\
\exists\neg\Diamond & \equiv & \exists\Box\neg \\
\exists\neg\Diamond\neg & \equiv & \exists\Box
\end{array}$$

$$\forall x(\Diamond Ax \to \neg\Box Bx) \equiv \forall x(\Diamond Ax \to \Diamond\neg Bx)$$

	$\Box$	$\Box\neg$	$\Diamond$	$\Diamond\neg$
$\forall$	$\forall\Box$	$\forall\Box\neg$	$\forall\Diamond$	$\forall\Diamond\neg$
$\forall\neg$	$\forall\neg\Box$	$\forall\neg\Box\neg$	$\forall\neg\Diamond$	$\forall\neg\Diamond\neg$
$\exists$	$\exists\Box$	$\exists\Box\neg$	$\exists\Diamond$	$\exists\Diamond\neg$
$\exists\neg$	$\exists\neg\Box$	$\exists\neg\Box\neg$	$\exists\neg\Diamond$	$\exists\neg\Diamond\neg$

- up to logical equivalence, we arrive at $\frac{4\times4}{2} = 8$ formulas
- these are exactly the formulas found in Buridan's modal octagon

- octagon $=$ square $\times$ square

- Buridan octagon = quantifier square $\times$ modality square

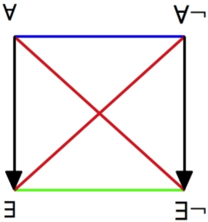
- take the Boolean closure of these components separately
- recall that the Boolean closure of a square is a JSB hexagon

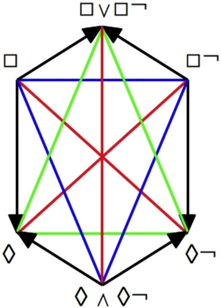
- quantifier square $\times$ modality hexagon
- quantifier hexagon $\times$ modality square
- quantifier hexagon $\times$ modality hexagon

- we will start by considering the first of these:

quantifier square $\times$ modality hexagon

A first extension of Buridan's octagon





	$\Box$	$\Box\neg$	$\Diamond$	$\Diamond\neg$	$\Box \vee \Box\neg$	$\Diamond \wedge \Diamond\neg$
$\forall$	$\forall\Box$	$\forall\Box\neg$	$\forall\Diamond$	$\forall\Diamond\neg$	$\forall(\Box \vee \Box\neg)$	$\forall(\Diamond \wedge \Diamond\neg)$
$\forall\neg$	$\forall\neg\Box$	$\forall\neg\Box\neg$	$\forall\neg\Diamond$	$\forall\neg\Diamond\neg$	$\forall\neg(\Box \vee \Box\neg)$	$\forall\neg(\Diamond \wedge \Diamond\neg)$
$\exists$	$\exists\Box$	$\exists\Box\neg$	$\exists\Diamond$	$\exists\Diamond\neg$	$\exists(\Box \vee \Box\neg)$	$\exists(\Diamond \wedge \Diamond\neg)$
$\exists\neg$	$\exists\neg\Box$	$\exists\neg\Box\neg$	$\exists\neg\Diamond$	$\exists\neg\Diamond\neg$	$\exists\neg(\Box \vee \Box\neg)$	$\exists\neg(\Diamond \wedge \Diamond\neg)$

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	$\Box$	$\Box\neg$	$\Diamond$	$\Diamond\neg$	$\Box \vee \Box\neg$	$\Diamond \wedge \Diamond\neg$
$\forall$	$\forall\Box$	$\forall\Box\neg$	$\forall\Diamond$	$\forall\Diamond\neg$	$\forall(\Box \vee \Box\neg)$	$\forall(\Diamond \wedge \Diamond\neg)$
$\forall\neg$	$\forall\neg\Box$	$\forall\neg\Box\neg$	$\forall\neg\Diamond$	$\forall\neg\Diamond\neg$	$\forall\neg(\Box \vee \Box\neg)$	$\forall\neg(\Diamond \wedge \Diamond\neg)$
$\exists$	$\exists\Box$	$\exists\Box\neg$	$\exists\Diamond$	$\exists\Diamond\neg$	$\exists(\Box \vee \Box\neg)$	$\exists(\Diamond \wedge \Diamond\neg)$
$\exists\neg$	$\exists\neg\Box$	$\exists\neg\Box\neg$	$\exists\neg\Diamond$	$\exists\neg\Diamond\neg$	$\exists\neg(\Box \vee \Box\neg)$	$\exists\neg(\Diamond \wedge \Diamond\neg)$

- note: $\forall(\Box \vee \Box\neg)$ should be read as: $\forall x(\Diamond Ax \rightarrow (\Box Bx \vee \Box\neg Bx))$
- 8 new formulas, but again pairwise equivalent:
  - $\forall\neg(\Box \vee \Box\neg) \equiv \forall(\Diamond \wedge \Diamond\neg)$ $\quad$ $\exists\neg(\Box \vee \Box\neg) \equiv \exists(\Diamond \wedge \Diamond\neg)$
  - $\forall\neg(\Diamond \wedge \Diamond\neg) \equiv \forall(\Box \vee \Box\neg)$ $\quad$ $\exists\neg(\Diamond \wedge \Diamond\neg) \equiv \exists(\Box \vee \Box\neg)$

	$\Box$	$\Box\neg$	$\Diamond$	$\Diamond\neg$	$\Box \vee \Box\neg$	$\Diamond \wedge \Diamond\neg$
$\forall$	$\forall\Box$	$\forall\Box\neg$	$\forall\Diamond$	$\forall\Diamond\neg$	$\forall(\Box \vee \Box\neg)$	$\forall(\Diamond \wedge \Diamond\neg)$
$\forall\neg$	$\forall\neg\Box$	$\forall\neg\Box\neg$	$\forall\neg\Diamond$	$\forall\neg\Diamond\neg$	$\forall\neg(\Box \vee \Box\neg)$	$\forall\neg(\Diamond \wedge \Diamond\neg)$
$\exists$	$\exists\Box$	$\exists\Box\neg$	$\exists\Diamond$	$\exists\Diamond\neg$	$\exists(\Box \vee \Box\neg)$	$\exists(\Diamond \wedge \Diamond\neg)$
$\exists\neg$	$\exists\neg\Box$	$\exists\neg\Box\neg$	$\exists\neg\Diamond$	$\exists\neg\Diamond\neg$	$\exists\neg(\Box \vee \Box\neg)$	$\exists\neg(\Diamond \wedge \Diamond\neg)$

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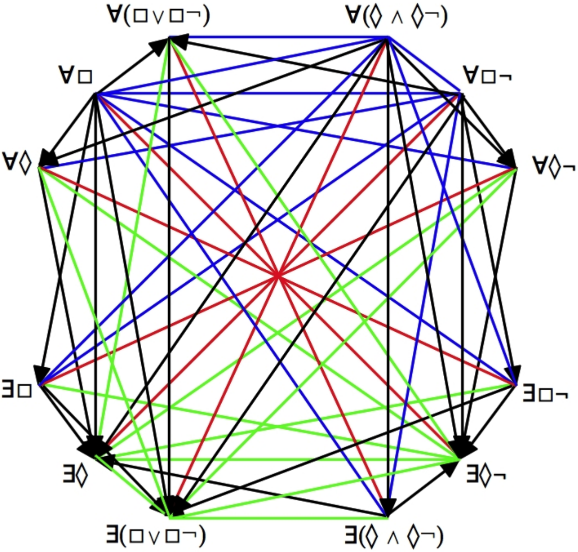
  - $\forall\neg(\Diamond \wedge \Diamond\neg) \equiv \forall(\Box \vee \Box\neg)$ $\quad$ $\exists\neg(\Diamond \wedge \Diamond\neg) \equiv \exists(\Box \vee \Box\neg)$
- up to logical equivalence, we arrive at $\frac{4\times 6}{2} = 12$ formulas
  $\Rightarrow$ Aristotelian dodecagon that extends Buridan's octagon
- more reasonable than the octagon's full Boolean closure $(8 < 12 \ll 62)$



- Buridan's works
  - contain the octagon
  - do not contain the dodecagon

- S. Read, 2015, John Buridan on Non-Contingency Syllogisms
  - identified the 12 formulas of the dodecagon
  - identified the Aristotelian relations of the dodecagon

- note: $\forall(\Box \vee \Box\neg)$ is not equivalent to $\forall\Box \vee \forall\Box\neg$
- Buridan: "this is true, 'No planet is contingently the moon', but this is false, 'Every planet is necessarily the moon or every planet necessarily fails to be the moon'." (Tractatus de Consequentiis)

no — contingently $= \forall\neg(\Diamond \wedge \Diamond\neg) \equiv \forall(\Box \vee \Box\neg) \not\equiv \forall\Box \vee \forall\Box\neg$

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- Buridan "had" a dodecagon (quantifier square $\times$ modality hexagon)
- S. Chatti, 2015, Les Carrés d'Avicenne
- Avicenna: ca. 980–1037 ($\pm$ 300 years before Buridan)
  - identified the 12 formulas of the dodecagon
  - identified the Aristotelian relations of the dodecagon
- but with temporal instead of modal operators

formula	Buridan	Avicenna
$\exists\Box$	some A are **necessarily** B	some A are **always** B
$\forall\Diamond$	all A are **possibly** B	all A are **sometimes** B

Buridan:	dodecagon	$=$	quantifier square	$\times$	modal hexagon
Avicenna:	dodecagon	$=$	quantifier square	$\times$	temporal hexagon

- the story so far:
  - Buridan: octagon = quantifier square $\times$ modality square
  - first extension: take Boolean closure of the second square

    $\Rightarrow$ dodecagon = quantifier square $\times$ modality hexagon

- now: second extension: take Boolean closure of the first square

  $\Rightarrow$ dodecagon = quantifier hexagon $\times$ modality square

  but also switch the roles of quantifiers and modalities

  $\Rightarrow$ dodecagon = modality hexagon $\times$ quantifier square

  (from de re modalities to de dicto modalities)

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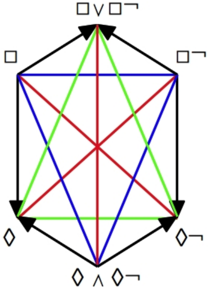
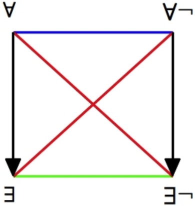




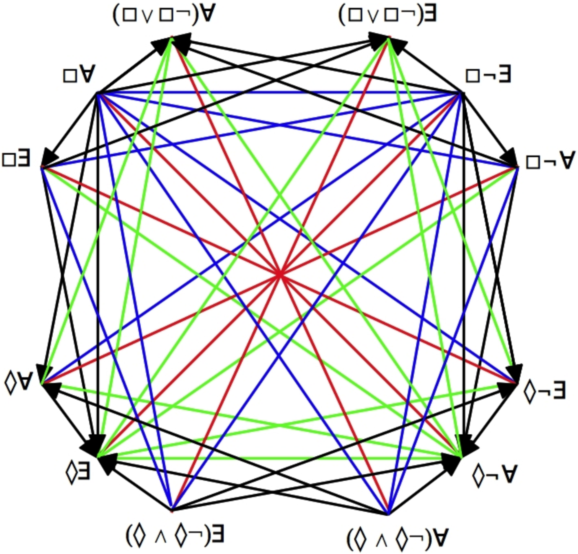
	$\forall$	$\forall\neg$	$\exists$	$\exists\neg$
$\Box$	$\Box\forall$	$\Box\forall\neg$	$\Box\exists$	$\Box\exists\neg$
$\Box\neg$	$\Box\neg\forall$	$\Box\neg\forall\neg$	$\Box\neg\exists$	$\Box\neg\exists\neg$
$\Diamond$	$\Diamond\forall$	$\Diamond\forall\neg$	$\Diamond\exists$	$\Diamond\exists\neg$
$\Diamond\neg$	$\Diamond\neg\forall$	$\Diamond\neg\forall\neg$	$\Diamond\neg\exists$	$\Diamond\neg\exists\neg$
$\Box \vee \Box\neg$	$(\Box \vee \Box\neg)\forall$	$(\Box \vee \Box\neg)\forall\neg$	$(\Box \vee \Box\neg)\exists$	$(\Box \vee \Box\neg)\exists\neg$
$\Diamond \wedge \Diamond\neg$	$(\Diamond \wedge \Diamond\neg)\forall$	$(\Diamond \wedge \Diamond\neg)\forall\neg$	$(\Diamond \wedge \Diamond\neg)\exists$	$(\Diamond \wedge \Diamond\neg)\exists\neg$

	$\forall$	$\forall\neg$	$\exists$	$\exists\neg$
$\Box$	$\Box\forall$	$\Box\forall\neg$	$\Box\exists$	$\Box\exists\neg$
$\Box\neg$	$\Box\neg\forall$	$\Box\neg\forall\neg$	$\Box\neg\exists$	$\Box\neg\exists\neg$
$\Diamond$	$\Diamond\forall$	$\Diamond\forall\neg$	$\Diamond\exists$	$\Diamond\exists\neg$
$\Diamond\neg$	$\Diamond\neg\forall$	$\Diamond\neg\forall\neg$	$\Diamond\neg\exists$	$\Diamond\neg\exists\neg$
$\Box \vee \Box\neg$	$(\Box \vee \Box\neg)\forall$	$(\Box \vee \Box\neg)\forall\neg$	$(\Box \vee \Box\neg)\exists$	$(\Box \vee \Box\neg)\exists\neg$
$\Diamond \wedge \Diamond\neg$	$(\Diamond \wedge \Diamond\neg)\forall$	$(\Diamond \wedge \Diamond\neg)\forall\neg$	$(\Diamond \wedge \Diamond\neg)\exists$	$(\Diamond \wedge \Diamond\neg)\exists\neg$

- note: $(\Box \vee \Box\neg)\forall$ should be read as: $\Box\forall \vee \Box\neg\forall$ ($\equiv \Box\forall \vee \Box\exists\neg$)
- $6 \times 4 = 24$ formulas, but again pairwise equivalent

	$\forall$	$\forall\neg$	$\exists$	$\exists\neg$
$\Box$	$\Box\forall$	$\Box\forall\neg$	$\Box\exists$	$\Box\exists\neg$
$\Box\neg$	$\Box\neg\forall$	$\Box\neg\forall\neg$	$\Box\neg\exists$	$\Box\neg\exists\neg$
$\Diamond$	$\Diamond\forall$	$\Diamond\forall\neg$	$\Diamond\exists$	$\Diamond\exists\neg$
$\Diamond\neg$	$\Diamond\neg\forall$	$\Diamond\neg\forall\neg$	$\Diamond\neg\exists$	$\Diamond\neg\exists\neg$
$\Box\vee\Box\neg$	$(\Box\vee\Box\neg)\forall$	$(\Box\vee\Box\neg)\forall\neg$	$(\Box\vee\Box\neg)\exists$	$(\Box\vee\Box\neg)\exists\neg$
$\Diamond\wedge\Diamond\neg$	$(\Diamond\wedge\Diamond\neg)\forall$	$(\Diamond\wedge\Diamond\neg)\forall\neg$	$(\Diamond\wedge\Diamond\neg)\exists$	$(\Diamond\wedge\Diamond\neg)\exists\neg$

- note: $(\Box\vee\Box\neg)\forall$ should be read as: $\Box\forall\vee\Box\neg\forall$ ($\equiv \Box\forall\vee\Box\exists\neg$)
- $6\times 4 = 24$ formulas, but again pairwise equivalent
- up to logical equivalence, we arrive at $\frac{6\times 4}{2} = 12$ formulas
  $\Rightarrow$ another Aristotelian dodecagon that extends Buridan's octagon



- S. Chatti, 2014, Avicenna on Possibility and Necessity
- Avicenna:
  - identified the 12 formulas of this second dodecagon
  - identified the Aristotelian relations holding between them

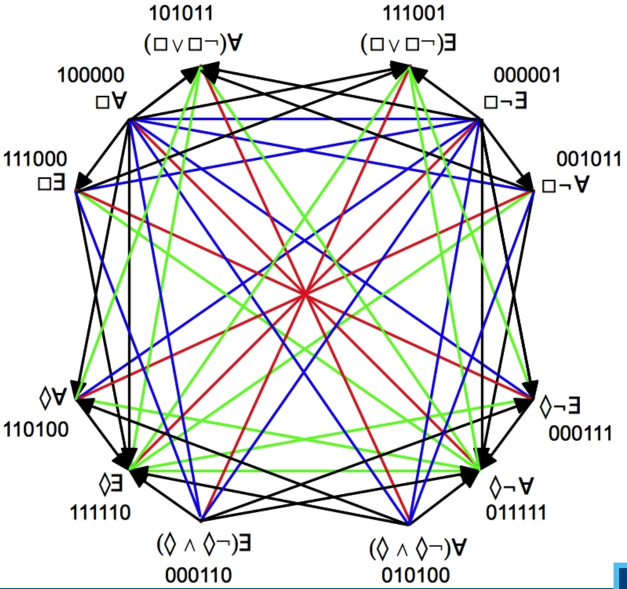
1. Some Preliminaries from Logical Geometry
2. Buridan's Aristotelian Diagrams
3. Avicenna's Aristotelian Diagrams
4. Bitstring Analysis
5. Conclusion

- recall: Buridan octagon $\Rightarrow$ bitstrings of length 6
- anchor formulas:

1.	$\forall\Box$	4.	$\exists\Box \wedge \exists\Box\neg$
2.	$\forall\Diamond \wedge \exists\Box \wedge \exists\Diamond\neg$	5.	$\forall\Diamond\neg \wedge \exists\Box\neg \wedge \exists\Diamond$
3.	$\forall\Diamond \wedge \forall\Diamond\neg$	6.	$\forall\Box\neg$

- second extension (hexagon $\times$ square) $\Rightarrow$ bitstrings of length 6
- anchor formulas: same as above
  (except that quantifiers and modalities should be switched)

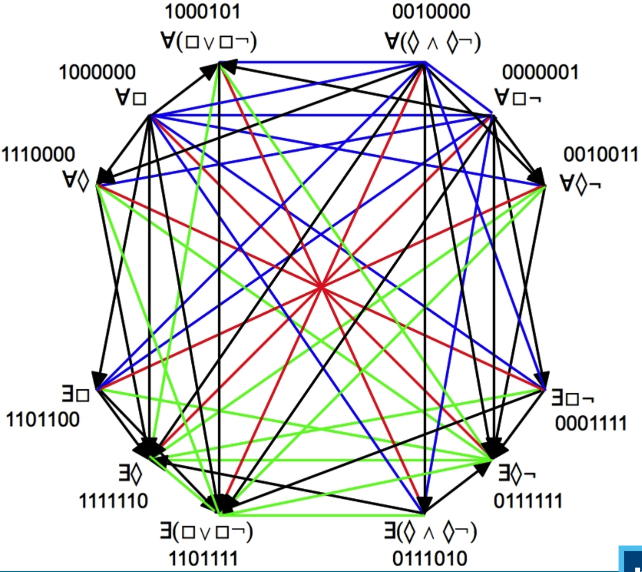
- this shows that the second extension of Buridan's octagon remains within the latter's Boolean closure


101011
(□ ∨ □¬)∀
111001
(□ ∨ □¬)∃
100000
□∀
000001
□¬∃
111000
□∃
001011
□¬∀
◊∀
110100
◊¬∃
000111
◊∃
111110
◊¬∀
011111
(◊ ∧ ◊¬)∃
000110
(◊ ∧ ◊¬)∀
010100

- first extension (square $\times$ hexagon) $\Rightarrow$ bitstrings of length 7
- anchor formulas:

1.	$\forall\Box$	4a.	$\exists\Box \wedge \exists\Box\neg \wedge \exists(\Diamond \wedge \Diamond\neg)$
2.	$\forall\Diamond \wedge \exists\Box \wedge \exists\Diamond\neg$	4b.	$\exists\Box \wedge \exists\Box\neg \wedge \forall(\Box \vee \Box\neg)$
3.	$\forall\Diamond \wedge \forall\Diamond\neg$	5.	$\forall\Diamond\neg \wedge \exists\Box\neg \wedge \exists\Diamond$
		6.	$\forall\Box\neg$

- same as for the octagon, except that 4 has been 'split' into 4a and 4b
- the first extension is essentially more complex than the original octagon
  - the first extension does not fit within the octagon's Boolean closure
  - Boolean closure of the octagon: $2^6 - 2 = 62$ formulas
  - Boolean closure of the first extension: $2^7 - 2 = 126$ formulas
- why so many additional formulas?
  - formulas where the quantifier does **not** distribute over the modality
  - cf. anchor formulas 4a and 4b



1000101
∀(□ ∨ □¬)
0010000
∀(◊ ∧ ◊¬)
1000000
∀□
0000001
∀□¬
1110000
∀◊
0010011
∀◊¬
∃□
1101100
∃□¬
0001111
∃◊
1111110
∃◊¬
0111111
∃(□ ∨ □¬)
1101111
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0111010

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- natural extension from a technical (and historical?) perspective:
  - take Boolean closure of both square components
  - so we get hexagon $\times$ hexagon $\Rightarrow \frac{6\times 6}{2} = 18$ formulas
  - e.g. "some but not all men are contingently philosophers"

- overview:

Buridan	8-gon	quantifier square	$\times$	modality square	6
"Al-Farabi"	8-gon	quantifier square	$\times$	modality square	6
"Buridan"	12-gon	quantifier square	$\times$	modality hexagon	7
"Avicenna"	12-gon	quantifier square	$\times$	temporal hexagon	7
"Avicenna"	12-gon	modality hexagon	$\times$	quantifier square	6
???	18-gon	quantifier hexagon	$\times$	modal hexagon	7

Thank you!

More info: www.logicalgeometry.org